Trigonometry and modelling 7D

1 a $3\cos\theta = 2\sin(\theta + 60°)$

$\Rightarrow\ 3\cos\theta = 2(\sin\theta\cos 60° + \cos\theta\sin 60°)$

$\Rightarrow\ 3\cos\theta = 2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = \sin\theta + \sqrt{3}\cos\theta$

$\Rightarrow\ \left(3-\sqrt{3}\right)\cos\theta = \sin\theta$

$\Rightarrow\ \tan\theta = 3-\sqrt{3} = 1.2679...\quad \left(\text{as } \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$

As $\tan\theta$ is positive, θ is in the first and third quadrants

$\theta = \tan^{-1}(1.2679),\ 180° + \tan^{-1}(1.2679)$

$\theta = 51.7°,\ 231.7°$

b $\sin(\theta + 30°) + 2\sin\theta = 0$

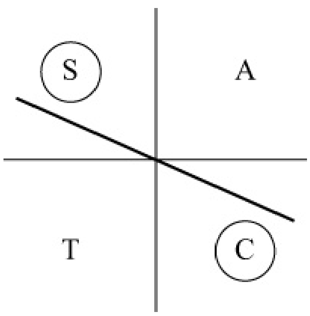
$\Rightarrow\ \sin\theta\cos 30° + \cos\theta\sin 30° + 2\sin\theta = 0$

$\Rightarrow\ \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + 2\sin\theta = 0$

$\Rightarrow\ \left(4+\sqrt{3}\right)\sin\theta = -\cos\theta$

$\Rightarrow\ \tan\theta = -\frac{1}{4+\sqrt{3}}$

As $\tan\theta$ is negative, θ is in the second and fourth quadrants



$\theta = \tan^{-1}\left(-\frac{1}{4+\sqrt{3}}\right) + 180°,\ \tan^{-1}\left(-\frac{1}{4+\sqrt{3}}\right) + 360°$

$\theta = 170.1°,\ 350.1°$

1 c $\cos(\theta+25°)+\sin(\theta+65°)=1$

$\Rightarrow \quad \cos\theta\cos 25° - \sin\theta\sin 25° + \sin\theta\cos 65° + \cos\theta\sin 65° = 1$

As $\sin(90-x)° = \cos x°$ and $\cos(90-x)° = \sin x°$

$\cos 25° = \sin 65°$ and $\sin 25° = \cos 65°$

So $\cos\theta\sin 65° - \sin\theta\cos 65° + \sin\theta\cos 65° + \cos\theta\sin 65° = 1$

$\Rightarrow \quad 2\cos\theta\sin 65° = 1$

$$\Rightarrow \quad \cos\theta = \frac{1}{2\sin 65°} = 0.55168...$$

$\theta = \cos^{-1}(0.55168),\ 360° - \cos^{-1}(0.55168)$

$\theta = \ 56.5°,\ 303.5°$

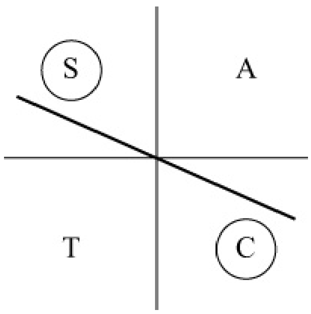
d $\cos\theta = \cos(\theta+60°)$

$$\Rightarrow \quad \cos\theta = \cos\theta\cos 60° - \sin\theta\sin 60° = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

$\Rightarrow \quad \cos\theta = -\sqrt{3}\sin\theta$

$$\Rightarrow \quad \tan\theta = -\frac{1}{\sqrt{3}} \qquad \left(\text{as } \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$$

As $\tan\theta$ is negative, θ is in the second and fourth quadrants



$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+180°,\ \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+360°$$

$\theta = 150.0°,\ 330.0°$

2 a $$\sin\left(\theta+\frac{\pi}{4}\right) \equiv \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}$$

$$\equiv \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta \equiv \frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$$

2 b $\dfrac{1}{\sqrt{2}}(\sin\theta+\cos\theta)=\dfrac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}$

Find all answers for $\theta+\dfrac{\pi}{4}$. As $0\le\theta\le 2\pi$ so $\dfrac{\pi}{4}\le\theta+\dfrac{\pi}{4}\le 2\pi+\dfrac{\pi}{4}$

$\theta+\dfrac{\pi}{4}=\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{9\pi}{4}$, so $\theta=0,\dfrac{\pi}{2},2\pi$

c As $\sin\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$

When $\sin\theta+\cos\theta=1$

$\sin\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}$

So $\theta=0,\dfrac{\pi}{2},2\pi$

3 a $\cos\theta\cos30°-\sin\theta\sin30°=0.5$

$\Rightarrow \cos(\theta+30°)=0.5$

$\Rightarrow \theta+30°=60°,\ 300°$

$\Rightarrow \theta=30°,\ 270°$

b $\cos\theta\cos30°-\sin\theta\sin30°\equiv\cos\theta\times\dfrac{\sqrt{3}}{2}-\sin\theta\times\dfrac{1}{2}$

So $\cos\theta\cos30°-\sin\theta\sin30°=\dfrac{1}{2}$ is identical to $\sqrt{3}\cos\theta-\sin\theta=1$

Solutions are same as (a), i.e. $30°,\ 270°$

4 a $3\sin(x-y)-\sin(x+y)=0$

$\Rightarrow 3\sin x\cos y-3\cos x\sin y-\sin x\cos y-\cos x\sin y=0$

$\Rightarrow 2\sin x\cos y=4\cos x\sin y$

$\Rightarrow \dfrac{2\sin x\cos y}{\cos x\cos y}=\dfrac{4\cos x\sin y}{\cos x\cos y}$

$\Rightarrow \dfrac{2\sin x}{\cos x}=\dfrac{4\sin y}{\cos y}$

$\Rightarrow 2\tan x=4\tan y$

b Put $y=45° \Rightarrow \tan x=2$

So $x=\tan^{-1}2,\ 180°+\tan^{-1}2$

$x=63.4°,\ 243.4°$ (1 d.p.)

5 a $\sin 2\theta = \sin\theta,\ 0 \le \theta \le 2\pi$

$\Rightarrow\ 2\sin\theta\cos\theta = \sin\theta$

$\Rightarrow\ 2\sin\theta\cos\theta - \sin\theta = 0$

$\Rightarrow\ \sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow\ \sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$

Solution set: $0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$

b $\cos 2\theta = 1 - \cos\theta,\ -180° < \theta \le 180°$

$\Rightarrow\ 2\cos^2\theta - 1 = 1 - \cos\theta$

$\Rightarrow\ 2\cos^2\theta + \cos\theta - 2 = 0$

$\Rightarrow\ \cos\theta = \frac{-1 \pm \sqrt{17}}{4}$ (using the quadratic formula)

As $\frac{-1-\sqrt{17}}{4} \le -1$, this gives the only solution as $\cos\theta = \frac{-1+\sqrt{17}}{4} = 0.78077...$

As $\cos\theta$ is positive, θ is in the first and fourth quadrants

Using a calculator $\cos^{-1} 0.78077 = 38.7°$ (1 d.p.)

Solutions are $\pm 38.7°$

c $3\cos 2\theta = 2\cos^2\theta,\quad 0 \le \theta < 360°$

$\Rightarrow\ 3(2\cos^2\theta - 1) = 2\cos^2\theta$

$\Rightarrow\ 6\cos^2\theta - 3 = 2\cos^2\theta$

$\Rightarrow\ 4\cos^2\theta = 3$

$\Rightarrow\ \cos^2\theta = \frac{3}{4}\ \Rightarrow\ \cos\theta = \pm\frac{\sqrt{3}}{2}$

θ will be in all four quadrants.

Solution set: 30°, 150°, 210°, 330°

d $\sin 4\theta = \cos 2\theta,\quad 0 \le \theta \le \pi$

$\Rightarrow\ 2\sin 2\theta\cos 2\theta = \cos 2\theta$

$\Rightarrow\ \cos 2\theta(2\sin 2\theta - 1) = 0$

$\Rightarrow\ \cos 2\theta = 0 \text{ or } \sin 2\theta = \frac{1}{2}$

$\cos 2\theta = 0$ in $0 \le 2\theta \le 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{2},\ \frac{3\pi}{2}\ \Rightarrow\ \theta = \frac{\pi}{4},\ \frac{3\pi}{4}$

$\sin 2\theta = \frac{1}{2}$ in $0 \le 2\theta \le 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{6},\ \frac{5\pi}{6}\ \Rightarrow\ \theta = \frac{\pi}{12},\ \frac{5\pi}{12}$

Solution set: $\frac{\pi}{12},\ \frac{\pi}{4},\ \frac{5\pi}{12},\ \frac{3\pi}{4}$

5 e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0, \quad 0 \le \theta \le 720°$

$\Rightarrow 3\left(1 - 2\sin^2\frac{\theta}{2}\right) - \sin\frac{\theta}{2} - 1 = 0$

$\Rightarrow 6\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - 2 = 0$

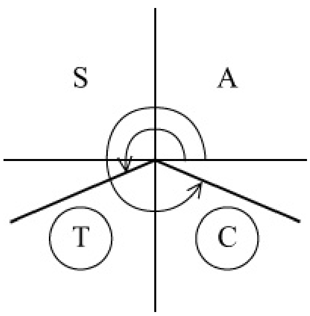
$\Rightarrow \left(3\sin\frac{\theta}{2} + 2\right)\left(2\sin\frac{\theta}{2} - 1\right) = 0$

$\Rightarrow \sin\frac{\theta}{2} = -\frac{2}{3}$ or $\sin\frac{\theta}{2} = \frac{1}{2}$

$\sin\frac{\theta}{2} = \frac{1}{2}$ in $0 \le \frac{\theta}{2} \le 360°$

$\Rightarrow \frac{\theta}{2} = 30°, 150° \Rightarrow \theta = 60°, 300°$

$\sin\frac{\theta}{2} = -\frac{2}{3}$ in $0 \le \frac{\theta}{2} \le 360°$



$\Rightarrow \frac{\theta}{2} = 180° - \sin^{-1}\left(-\frac{2}{3}\right), 360° + \sin^{-1}\left(-\frac{2}{3}\right) = 221.8°, 318.2°$ (1 d.p.)

$\Rightarrow \theta = 443.6°, 636.4°$

Solution set: $60°, 300°, 443.6°, 636.4°$

f $\cos^2\theta - \sin 2\theta = \sin^2\theta, \qquad 0 \le \theta \le \pi$

$\Rightarrow \cos^2\theta - \sin^2\theta = \sin 2\theta$

$\Rightarrow \cos 2\theta = \sin 2\theta$

$\Rightarrow \tan 2\theta = 1$ (divide both sides by $\cos 2\theta$)

$\tan 2\theta = 1$ in $0 \le 2\theta \le 2\pi$

$\Rightarrow 2\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}$

5 g $2\sin\theta = \sec\theta, \quad 0 \le \theta \le 2\pi$

$\Rightarrow \; 2\sin\theta = \dfrac{1}{\cos\theta}$

$\Rightarrow \; 2\sin\theta\cos\theta = 1$

$\Rightarrow \; \sin 2\theta = 1$

$\sin 2\theta = 1$ in $0 \le 2\theta \le 4\pi$

$\Rightarrow \; 2\theta = \dfrac{\pi}{2}, \; \dfrac{5\pi}{2}$

$\Rightarrow \; \theta = \dfrac{\pi}{4}, \; \dfrac{5\pi}{4}$

h $2\sin 2\theta = 3\tan\theta, \quad 0 \le \theta < 360°$

$\Rightarrow \; 4\sin\theta\cos\theta = \dfrac{3\sin\theta}{\cos\theta}$

$\Rightarrow \; 4\sin\theta\cos^2\theta = 3\sin\theta$

$\Rightarrow \; \sin\theta(4\cos^2\theta - 3) = 0$

$\Rightarrow \; \sin\theta = 0$ or $\cos^2\theta = \dfrac{3}{4}$

$\sin\theta = 0 \; \Rightarrow \; \theta = 0°, \; 180°$

$\cos^2\theta = \dfrac{3}{4} \; \Rightarrow \cos\theta = \pm\dfrac{\sqrt{3}}{2} \; \Rightarrow \; \theta = \; 30°, \; 150°, \; 210°, \; 330°$

Solution set: $0°, \; 30°, \; 150°, \; 180°, \; 210°, \; 330°$

5 i $2\tan\theta = \sqrt{3}(1-\tan\theta)(1+\tan\theta), \quad 0 \le \theta \le 2\pi$

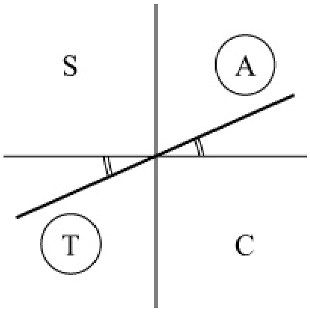
$\Rightarrow \ 2\tan\theta = \sqrt{3}(1-\tan^2\theta)$

$\Rightarrow \ \sqrt{3}\tan^2\theta + 2\tan\theta - \sqrt{3} = 0$

$\Rightarrow \ (\sqrt{3}\tan\theta - 1)(\tan\theta + \sqrt{3}) = 0$

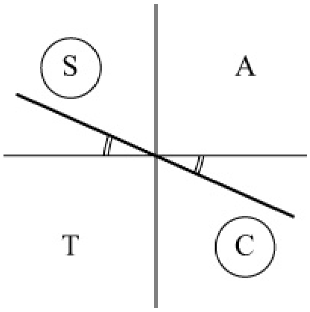
$\Rightarrow \ \tan\theta = \dfrac{1}{\sqrt{3}}$ or $\tan\theta = -\sqrt{3}$

$\tan\theta = \dfrac{1}{\sqrt{3}}, \ 0 \le \theta \le 2\pi$



$\Rightarrow \ \theta = \tan^{-1}\dfrac{1}{\sqrt{3}}, \ \pi + \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}, \ \dfrac{7\pi}{6}$

$\tan\theta = -\sqrt{3}, \quad 0 \le \theta \le 2\pi$



$\Rightarrow \ \theta = \pi + \tan^{-1}\left(-\sqrt{3}\right), \ 2\pi + \tan^{-1}\left(-\sqrt{3}\right) = \dfrac{2\pi}{3}, \ \dfrac{5\pi}{3}$

Solution set: $\dfrac{\pi}{6}, \ \dfrac{2\pi}{3}, \ \dfrac{7\pi}{6}, \ \dfrac{5\pi}{3}$

5 j $\sin^2\theta = 2\sin 2\theta, \quad -180° < \theta \leq 180°$

$\Rightarrow \sin^2\theta = 4\sin\theta\cos\theta$

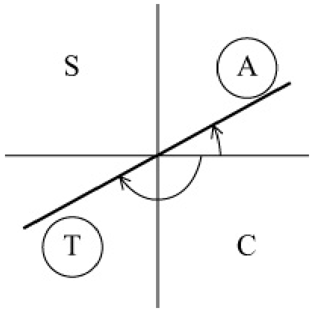
$\Rightarrow \sin\theta(\sin\theta - 4\cos\theta) = 0$

$\Rightarrow \sin\theta = 0 \text{ or } \sin\theta = 4\cos\theta$

$\Rightarrow \sin\theta = 0 \text{ or } \tan\theta = 4$

$\sin\theta = 0 \Rightarrow \theta = 0°, 180°$

$\tan\theta = 4 \Rightarrow \theta = \tan^{-1}4, \; -180° + \tan^{-1}4 = 76.0°, \; -104.0°$ (1 d.p.)



Solution set: $-104.0°, \; 0°, \; 76.0°$

k $4\tan\theta = \tan 2\theta, \quad 0 \leq \theta < 360°$

$\Rightarrow 4\tan\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow 2\tan\theta(1-\tan^2\theta) = \tan\theta$

$\Rightarrow \tan\theta(2 - 2\tan^2\theta - 1) = 0$

$\Rightarrow \tan\theta(1 - 2\tan^2\theta) = 0$

$\Rightarrow \tan\theta = 0 \; or \; \tan\theta = \pm\sqrt{\dfrac{1}{2}}$

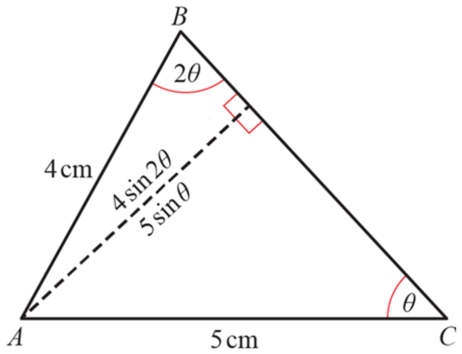
$\tan\theta = 0 \Rightarrow \theta = 0°, 180°$

$\tan\theta = \pm\sqrt{\dfrac{1}{2}}$

$\Rightarrow \theta = 35.3°, \; 144.7°, \; 215.3°, \; 324.7°$

Solution set: $0°, \; 35.3°, \; 144.7°, \; 180°, \; 215.3°, \; 324.7°$

6 Sketch ΔABC



$4\sin 2\theta = 5\sin\theta$

$\Rightarrow 8\sin\theta\cos\theta = 5\sin\theta$

$\Rightarrow 8\sin\theta\cos\theta - 5\sin\theta = 0$

$\Rightarrow \sin\theta(8\cos\theta - 5) = 0$

$\Rightarrow \sin\theta = 0 \text{ or } \cos\theta = \frac{5}{8}$

As ABC is a triangle, $0 < \theta < 90°$, so $\theta = 0°$ or $180°$ are not possible solutions.

So $\theta = \cos^{-1}\left(\frac{5}{8}\right) = 51.3°$ (1 d.p.)

7 **a** As $5\sin 2\theta = 10\sin\theta\cos\theta$

$5\sin 2\theta + 4\sin\theta = 10\sin\theta\cos\theta + 4\sin\theta = 0$

$2\sin\theta(5\cos\theta + 2) = 0$

So $a = 2$, $b = 5$ and $c = 2$

b $2\sin\theta(5\cos\theta + 2) = 0, \quad 0 \le \theta < 360°$

$\Rightarrow \sin\theta = 0 \text{ or } \cos\theta = -\frac{2}{5}$

$\sin\theta = 0 \ \Rightarrow \theta = 0°,\ 180°$

$\cos\ \theta = -\frac{2}{5} \ \Rightarrow \theta = \cos^{-1}\left(-\frac{2}{5}\right),\ 360° - \cos^{-1}\left(-\frac{2}{5}\right) = 113.6°,\ 246.4°$ (1 d.p.)

Solution set: $\theta = 0°,\ 113.6°,\ 180°,\ 246.4°$

8 **a** As $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$

$\sin 2\theta + \cos 2\theta = 1$

$\Rightarrow 2\sin\theta\cos\theta + (1 - 2\sin^2\theta) = 1$

$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$

$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$

8 b $2\sin\theta(\cos\theta - \sin\theta) = 0, \quad 0 \le \theta < 360°$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \sin\theta$

$\sin\theta = 0 \Rightarrow \theta = 0°, 180°$

$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°, 225°$

Solution set: $\theta = 0°, 45°, 180°, 225°$

9 a
$$\begin{aligned}\text{LHS} &\equiv (\cos 2\theta - \sin 2\theta)^2\\ &\equiv \cos^2 2\theta - 2\sin 2\theta\cos 2\theta + \sin^2 2\theta\\ &\equiv (\cos^2 2\theta + \sin^2 2\theta) - (2\sin 2\theta\cos 2\theta)\\ &\equiv 1 - \sin 4\theta \qquad (\sin^2 A + \cos^2 A \equiv 1,\ \sin 2A \equiv 2\sin A\cos A)\\ &\equiv \text{RHS}\end{aligned}$$

b You can use $(\cos 2\theta - \sin 2\theta)^2 = \frac{1}{2}$ but this also solves the equation

$\cos 2\theta - \sin 2\theta = -\frac{1}{\sqrt{2}}$

so you need to check your final answers.

As $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$

$\Rightarrow \frac{1}{2} = 1 - \sin 4\theta$

$\Rightarrow \sin 4\theta = \frac{1}{2}$

$0 \le \theta < \pi$, so $0 \le 4\theta < 4\pi$

$\Rightarrow 4\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}$

Checking these values in $\cos 2\theta - \sin 2\theta = \frac{1}{\sqrt{2}}$ eliminates $\frac{5\pi}{24}, \frac{13\pi}{24}$

which apply to $\cos 2\theta - \sin 2\theta = -\frac{1}{\sqrt{2}}$

Solutions are $\frac{\pi}{24}, \frac{17\pi}{24}$

10 a i RHS
$$\equiv \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$
$$\equiv \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$
$$\equiv \frac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\times\cos^2\frac{\theta}{2}$$
$$\equiv 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$\equiv \sin\theta \qquad (\sin 2A = 2\sin A\cos A)$$
$$\equiv \text{LHS}$$

ii RHS
$$\equiv \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$
$$\equiv \frac{1-\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$
$$\equiv \cos^2\tfrac{\theta}{2}\left(1-\tan^2\tfrac{\theta}{2}\right)$$
$$\equiv \cos^2\tfrac{\theta}{2}-\sin^2\tfrac{\theta}{2} \qquad \left(\tan^2\tfrac{\theta}{2}=\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)$$
$$\equiv \cos\theta \qquad (\cos 2A = \cos^2 A - \sin^2 A)$$
$$\equiv \text{LHS}$$

b Let $\tan\dfrac{\theta}{2}=t$

i $\sin\theta + 2\cos\theta = 1$
$$\Rightarrow \frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}=1$$
$$\Rightarrow 2t+2-2t^2=1+t^2$$
$$\Rightarrow 3t^2-2t-1=0$$
$$\Rightarrow (3t+1)(t-1)=0$$
$$\Rightarrow \tan\frac{\theta}{2}=-\frac{1}{3} \text{ or } \tan\frac{\theta}{2}=1, \quad 0\le\frac{\theta}{2}\le 180°$$
$$\tan\frac{\theta}{2}=1 \Rightarrow \frac{\theta}{2}=45° \Rightarrow \theta=90°$$
$$\tan\frac{\theta}{2}=-\frac{1}{3} \Rightarrow \frac{\theta}{2}=161.56° \Rightarrow \theta=323.1° \text{ (1 d.p.)}$$
Solution set: $90°,\ 323.1°$

10 b ii $3\cos\theta - 4\sin\theta = 2$

$\Rightarrow \dfrac{3(1-t^2)}{1+t^2} - \dfrac{4\times 2t}{1+t^2} = 2$

$\Rightarrow 3(1-t^2) - 8t = 2(1+t^2)$

$\Rightarrow 5t^2 + 8t - 1 = 0$

$\Rightarrow t = \dfrac{-8 \pm \sqrt{84}}{10}$

For $\tan\dfrac{\theta}{2} = \dfrac{-8+\sqrt{84}}{10} \qquad 0 \le \dfrac{\theta}{2} \le 180°$

$\dfrac{\theta}{2} = 6.65° \Rightarrow \theta = 13.3°$ (1 d.p.)

For $\tan\dfrac{\theta}{2} = \dfrac{-8-\sqrt{84}}{10} \qquad 0 \le \dfrac{\theta}{2} \le 180°$

$\dfrac{\theta}{2} = 120.2° \Rightarrow \theta = 240.4°$ (1 d.p.)

Solution set: 13.3°, 240.4°

11 a RHS $\equiv 1 + 2\cos 2x$

$\equiv 1 + 2(\cos^2 x - \sin^2 x)$

$\equiv 1 + 2\cos^2 x - 2\sin^2 x$

$\equiv \cos^2 x + \sin^2 x + 2\cos^2 x - 2\sin^2 x \qquad$ (using $\sin^2 x + \cos^2 x \equiv 1$)

$\equiv 3\cos^2 x - \sin^2 x$

$\equiv$ LHS

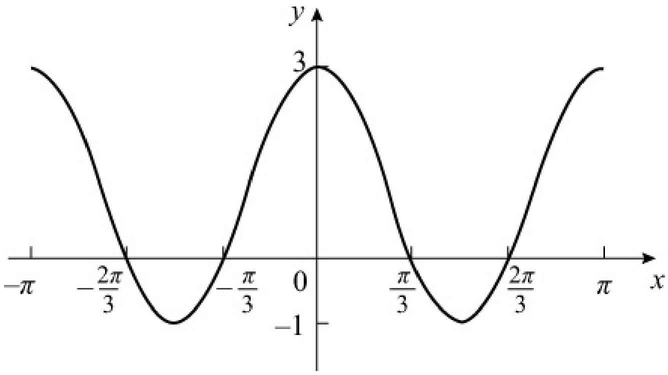
11 b $y = 3\cos^2 x - \sin^2 x$ is the same as $y = 1 + 2\cos 2x$

Using your work on transformations, this curve is the result of

(i) stretching $y = \cos x$ by scale factor $\frac{1}{2}$ in the x direction, then

(ii) stretching the result by scale factor 2 in the y direction, then

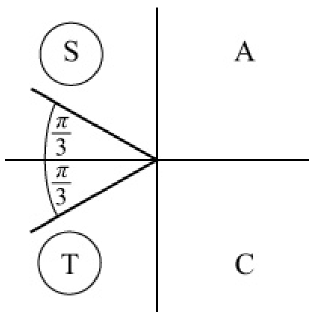
(iii) translating by 1 in the positive y direction.



The curve crosses y-axis at (0, 3). It crosses x-axis where $y = 0$

i.e. where $1 + 2\cos 2x = 0 \qquad -\pi \le x \le \pi$

$$\Rightarrow \cos 2x = -\frac{1}{2} \quad -2\pi \le 2x \le 2\pi$$



So $2x = -\frac{4\pi}{3}, \ -\frac{2\pi}{3}, \ \frac{2\pi}{3}, \ \frac{4\pi}{3}$

$$\Rightarrow x = -\frac{2\pi}{3}, \ -\frac{\pi}{3}, \ \frac{\pi}{3}, \ \frac{2\pi}{3}$$

The curve meets the x-axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 a $\cos^2 \frac{\theta}{2} = \frac{1 + \cos\theta}{2}, \ \sin^2 \frac{\theta}{2} = \frac{1 - \cos\theta}{2}$

So $2\cos^2 \frac{\theta}{2} - 4\sin^2 \frac{\theta}{2} = (1 + \cos\theta) - 2(1 - \cos\theta) = 3\cos\theta - 1$

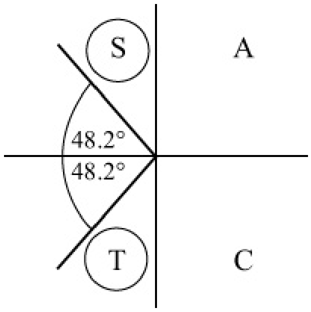
12 b $3\cos\theta - 1 = -3, \quad 0 \le \theta < 360°$

$\Rightarrow \ 3\cos\theta = -2$

$\Rightarrow \ \cos\theta = -\frac{2}{3}$

As $\cos\theta$ is negative, θ is in second and third quadrants.

Calculator value is $\cos^{-1}\left(-\frac{2}{3}\right) = 131.8°$ (1 d.p.)



Solutions are $131.8°,\ 360° - 131.8° = 131.8°,\ 228.2°$ (1 d.p.)

13 a As $\sin^2 A + \cos^2 A \equiv 1$ so $(\sin^2 A + \cos^2 A)^2 \equiv 1$

$\Rightarrow \ \sin^4 A + \cos^4 A + 2\sin^2 A\cos^2 A \equiv 1$

$\Rightarrow \ \sin^4 A + \cos^4 A \equiv 1 - 2\sin^2 A\cos^2 A$

$$\equiv 1 - \frac{1}{2}(4\sin^2 A\cos^2 A)$$

$$\equiv 1 - \frac{1}{2}\left((2\sin A\cos A)^2\right)$$

$$\equiv 1 - \frac{1}{2}\sin^2 2A$$

$$\equiv \frac{1}{2}(2 - \sin^2 2A)$$

b As $\cos 2A \equiv 1 - 2\sin^2 A$ so $\cos 4A \equiv 1 - 2\sin^2 2A$ so $\sin^2 2A \equiv \frac{1-\cos 4A}{2}$

$\Rightarrow$ from (a) $\sin^4 A + \cos^4 A \equiv \frac{1}{2}\left(2 - \frac{1-\cos 4A}{2}\right) \equiv \frac{1}{2}\left(\frac{4-1+\cos 4A}{2}\right) \equiv \frac{1}{4}(3+\cos 4A)$

13 c Using part (b)

$8\sin^4\theta + 8\cos^4\theta = 7$

$\Rightarrow\ 8\times\frac{1}{4}(3+\cos 4\theta) = 7$

$\Rightarrow\ 3+\cos 4\theta = \frac{7}{2}$

$\Rightarrow\ \cos 4\theta = \frac{1}{2}$

Solve $\cos 4\theta = \frac{1}{2}$ in $0 < 4\theta < 4\pi$

$\Rightarrow\ 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\Rightarrow\ \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

14 a $\cos 3\theta \equiv \cos(2\theta+\theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$\equiv \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta$

b $6\cos\theta - 8\cos^3\theta + 1 = 0$, $\quad 0 < \theta < \pi$

$\Rightarrow 1 = 8\cos^3\theta - 6\cos\theta$

$\Rightarrow 4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$\Rightarrow \cos 3\theta = \frac{1}{2}$, $\quad 0 < 3\theta < 3\pi$ using the result from part (a)

So $3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$