Trigonometry and Modelling 7F

1 **a** LHS $\equiv \dfrac{\cos 2A}{\cos A + \sin A}$

$\equiv \dfrac{\cos^2 A - \sin^2 A}{\cos A + \sin A}$

$\equiv \dfrac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A + \sin A}$

$\equiv \cos A - \sin A \equiv$ RHS

b LHS $\equiv \dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A}$

$\equiv \dfrac{\sin B \cos A - \cos B \sin A}{\sin A \cos A}$

$\equiv \dfrac{\sin(B-A)}{\frac{1}{2}(2\sin A \cos A)}$

$\equiv \dfrac{2\sin(B-A)}{\sin 2A}$

$\equiv 2\operatorname{cosec} 2A \sin(B-A) \equiv$ RHS

c LHS $\equiv \dfrac{1-\cos 2\theta}{\sin 2\theta}$

$\equiv \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$

$\equiv \dfrac{2\sin^2\theta}{2\sin\theta\cos\theta}$

$\equiv \dfrac{\sin\theta}{\cos\theta}$

$\equiv \tan\theta \equiv$ RHS

d LHS $\equiv \dfrac{\sec^2\theta}{1-\tan^2\theta}$

$\equiv \dfrac{1}{\cos^2\theta(1-\tan^2\theta)}$

$\equiv \dfrac{1}{\cos^2\theta - \sin^2\theta}$ $\left(\text{as } \tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}\right)$

$\equiv \dfrac{1}{\cos 2\theta}$

$\equiv \sec 2\theta \equiv$ RHS

1 e $\text{LHS} \equiv 2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)$

$\equiv 2\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$

$\equiv \sin 2\theta \quad (\text{since } \sin^2\theta + \cos^2\theta \equiv 1)$

$\equiv \text{RHS}$

f $\text{LHS} \equiv \dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta}$

$\equiv \dfrac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$

$\equiv \dfrac{\sin(3\theta - \theta)}{\frac{1}{2}\sin 2\theta}$

$\equiv \dfrac{\sin 2\theta}{\frac{1}{2}\sin 2\theta}$

$\equiv 2 \equiv \text{RHS}$

g $\text{LHS} \equiv \operatorname{cosec}\theta - 2\cot 2\theta\cos\theta$

$\equiv \operatorname{cosec}\theta - 2\dfrac{\cos 2\theta}{\sin 2\theta}\cos\theta$

$\equiv \operatorname{cosec}\theta - \dfrac{2\cos 2\theta\cos\theta}{2\sin\theta\cos\theta}$

$\equiv \dfrac{1}{\sin\theta} - \dfrac{\cos 2\theta}{\sin\theta}$

$\equiv \dfrac{1 - \cos 2\theta}{\sin\theta}$

$\equiv \dfrac{1 - (1 - 2\sin^2\theta)}{\sin\theta}$

$\equiv \dfrac{2\sin^2\theta}{\sin\theta}$

$\equiv 2\sin\theta \equiv \text{RHS}$

h $\text{LHS} \equiv \dfrac{\sec\theta - 1}{\sec\theta + 1}$

$\equiv \dfrac{\frac{1}{\cos\theta} - 1}{\frac{1}{\cos\theta} + 1}$

$\equiv \dfrac{1 - \cos\theta}{1 + \cos\theta}$

$\equiv \dfrac{1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)}{1 + (2\cos^2\frac{\theta}{2} - 1)}$

$\equiv \dfrac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$

$\equiv \tan^2\dfrac{\theta}{2} \equiv \text{RHS}$

1 i LHS $\equiv \tan\left(\frac{\pi}{4} - x\right)$

$\equiv \dfrac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$

$\equiv \dfrac{1 - \tan x}{1 + \tan x}$

$\equiv \dfrac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}$

$\equiv \dfrac{\cos x - \sin x}{\cos x + \sin x}$

$\equiv \dfrac{\cos^2 x + \sin^2 x - 2\sin x\cos x}{\cos^2 x - \sin^2 x}$ (multiply 'top and bottom' by $\cos x - \sin x$)

$\equiv \dfrac{1 - \sin 2x}{\cos 2x} \equiv$ RHS

2 a LHS $\equiv \sin(A + 60°) + \sin(A - 60°)$

$\equiv \sin A\cos 60° + \cos A\sin 60° + \sin A\cos 60° - \cos A\sin 60°$

$\equiv 2\sin A\cos 60°$

$\equiv \sin A \quad \left(\text{since } \cos 60° = \frac{1}{2}\right)$

$\equiv$ RHS

b LHS $\equiv \dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B}$

$\equiv \dfrac{\cos A\cos B - \sin A\sin B}{\sin B\cos B}$

$\equiv \dfrac{\cos(A + B)}{\sin B\cos B}$

$\equiv$ RHS

c LHS $\equiv \dfrac{\sin(x + y)}{\cos x\cos y}$

$\equiv \dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y}$

$\equiv \dfrac{\sin x\cos y}{\cos x\cos y} + \dfrac{\cos x\sin y}{\cos x\cos y}$

$\equiv \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}$

$\equiv \tan x + \tan y$

$\equiv$ RHS

2 d LHS $\equiv \dfrac{\cos(x+y)}{\sin x \sin y} + 1$

$\equiv \dfrac{\cos(x+y) + \sin x \sin y}{\sin x \sin y}$

$\equiv \dfrac{\cos x \cos y - \sin x \sin y + \sin x \sin y}{\sin x \sin y}$

$\equiv \dfrac{\cos x \cos y}{\sin x \sin y}$

$\equiv \cot x \cot y$

$\equiv$ RHS

e LHS $\equiv \cos\left(\theta + \dfrac{\pi}{3}\right) + \sqrt{3}\sin\theta$

$\equiv \cos\theta\cos\dfrac{\pi}{3} - \sin\theta\sin\dfrac{\pi}{3} + \sqrt{3}\sin\theta$

$\equiv \dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta + \sqrt{3}\sin\theta$

$\equiv \dfrac{\sqrt{3}}{2}\sin\theta + \dfrac{1}{2}\cos\theta$

$\equiv \sin\theta\cos\dfrac{\pi}{6} + \cos\theta\sin\dfrac{\pi}{6} \quad \left(\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}, \sin\dfrac{\pi}{6} = \dfrac{1}{2}\right)$

$\equiv \sin\left(\theta + \dfrac{\pi}{6}\right) \quad (\sin(A+B))$

$\equiv$ RHS

f LHS $\equiv \cot(A+B) \equiv \dfrac{\cos(A+B)}{\sin(A+B)}$

$\equiv \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

$\equiv \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}}$ (dividing top and bottom by $\sin A \sin B$)

$\equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B} \equiv$ RHS

2 g LHS $\equiv \sin^2(45° + \theta) + \sin^2(45° - \theta)$

$\equiv (\sin 45° \cos\theta + \cos 45° \sin\theta)^2 + (\sin 45° \cos\theta - \cos 45° \sin\theta)^2$

$\equiv (\sin 45° \cos\theta + \sin 45° \sin\theta)^2 + (\sin 45° \cos\theta - \sin 45° \sin\theta)^2$ (as $\sin 45° = \cos 45°$)

$\equiv (\sin 45°)^2 \left((\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2\right)$

$\equiv \frac{1}{2}(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta)$

$\equiv \frac{1}{2}\left(2\left(\sin^2\theta + \cos^2\theta\right)\right)$

$\equiv \frac{1}{2} \times 2$ $(\sin^2\theta + \cos^2\theta \equiv 1)$

$\equiv 1$

$\equiv$ RHS

Alternatively as $\sin(90° - x°) \equiv \cos x°$, if $x = 45° + \theta°$ then $\sin(45° - \theta°) \equiv \cos(45° + \theta°)$ and original LHS becomes $\sin^2(45 + \theta)° + \cos^2(45 + \theta)°$, which $= 1$

h LHS $\equiv \cos(A + B)\cos(A - B)$

$\equiv (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$\equiv \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$\equiv \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$\equiv \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B$

$\equiv \cos^2 A - \sin^2 B$

$\equiv$ RHS

3 a LHS $\equiv \tan\theta + \cot\theta$

$\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$

$\equiv \frac{2}{2\sin\theta\cos\theta}$ $(\sin^2\theta + \cos^2\theta \equiv 1)$

$\equiv \frac{2}{\sin 2\theta}$

$\equiv 2\operatorname{cosec} 2\theta \equiv$ RHS

b Use $\theta = 75°$

$\Rightarrow \tan 75° + \cot 75° = 2\operatorname{cosec} 150° = 2 \times \frac{1}{\sin 150°} = 2 \times \frac{1}{\frac{1}{2}} = 4$

4 a $\sin 3\theta \equiv \sin(2\theta+\theta) \equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$\equiv (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$

$\equiv 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$\equiv 3\sin\theta\cos^2\theta - \sin^3\theta$

b $\cos 3\theta \equiv \cos(2\theta+\theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$\equiv \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

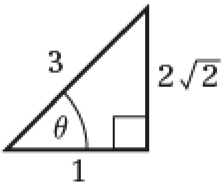
c $\tan 3\theta \equiv \dfrac{\sin 3\theta}{\cos 3\theta} \equiv \dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$

$$\equiv \dfrac{\dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta - 3\sin^2\theta\cos\theta}{\cos^3\theta}}$$

$$\equiv \dfrac{\dfrac{3\sin\theta}{\cos\theta} - \dfrac{\sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta}{\cos^3\theta} - \dfrac{3\sin^2\theta}{\cos^2\theta}}$$

$$\equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

d Sketch the right-angled triangle containing θ



This shows $\tan\theta = 2\sqrt{2}$

So $\tan 3\theta = \dfrac{3\left(2\sqrt{2}\right) - \left(2\sqrt{2}\right)^3}{1 - 3\left(2\sqrt{2}\right)^2} = \dfrac{6\sqrt{2} - 16\sqrt{2}}{1 - 24} = \dfrac{-10\sqrt{2}}{-23} = \dfrac{10\sqrt{2}}{23}$

5 a i Using $\cos 2A \equiv 2\cos^2 A - 1$ with $A = \dfrac{x}{2}$

$\Rightarrow \quad \cos x \equiv 2\cos^2\dfrac{x}{2} - 1$

$\Rightarrow \quad 2\cos^2\dfrac{x}{2} \equiv 1 + \cos x$

$\Rightarrow \quad \cos^2\dfrac{x}{2} \equiv \dfrac{1+\cos x}{2}$

5 a ii Using $\cos 2A \equiv 1 - 2\sin^2 A$

$$\Rightarrow \quad \cos x \equiv 1 - 2\sin^2 \frac{x}{2}$$

$$\Rightarrow \quad 2\sin^2 \frac{x}{2} \equiv 1 - \cos x$$

$$\Rightarrow \quad \sin^2 \frac{x}{2} \equiv \frac{1-\cos x}{2}$$

b i Using (a) (i) $\cos^2 \frac{\theta}{2} = \frac{1+\cos\theta}{2} = \frac{1.6}{2} = 0.8 = \frac{4}{5}$

$$\Rightarrow \quad \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \quad \left(\text{as } \frac{\theta}{2} \text{ acute}\right)$$

ii Using (a) (ii) $\sin^2 \frac{\theta}{2} = \frac{1-\cos\theta}{2} = \frac{0.4}{2} = 0.2 = \frac{1}{5}$

$$\Rightarrow \quad \sin\frac{\theta}{2} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$

iii $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sqrt{5}}{5} \times \frac{5}{2\sqrt{5}} = \frac{1}{2}$

c Using (a) (i) and squaring

$$\cos^4 \frac{A}{2} \equiv \left(\frac{1+\cos A}{2}\right)^2 \equiv \frac{1+2\cos A+\cos^2 A}{4}$$

but using $\cos 2A \equiv 2\cos^2 A - 1$ gives

$$\cos^2 A \equiv \frac{1}{2}(1+\cos 2A)$$

$$\text{So } \cos^4 \frac{A}{2} \equiv \frac{1+2\cos A+\frac{1}{2}(1+\cos 2A)}{4} \equiv \frac{2+4\cos A+1+\cos 2A}{8} \equiv \frac{3+4\cos A+\cos 2A}{8}$$

6 LHS $\equiv \cos^4\theta \equiv (\cos^2\theta)^2 \equiv \left(\frac{1+\cos 2\theta}{2}\right)^2$

$$\equiv \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta)$$

$$\equiv \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right)$$

$$\equiv \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{8}+\frac{1}{8}\cos 4\theta$$

$$\equiv \frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta \equiv \text{ RHS}$$

7 $\sin^2(x+y)-\sin^2(x-y)\equiv[\sin(x+y)+\sin(x-y)][\sin(x+y)-\sin(x-y)]$

$\equiv[\sin x\cos y+\cos x\sin y+\sin x\cos y-\cos x\sin y][\sin x\cos y+\cos x\sin y-(\sin x\cos y-\cos x\sin y)]$

$\equiv[2\sin x\cos y][2\cos x\sin y]$

$\equiv[2\sin x\cos x][2\cos y\sin y]$

$\equiv\sin 2x\sin 2y$

8 Let $\cos 2\theta-\sqrt{3}\sin 2\theta\equiv R\cos(2\theta+\alpha)\equiv R\cos 2\theta\cos\alpha-R\sin 2\theta\sin\alpha$

Compare $\cos 2\theta: R\cos\alpha=1$ (1)

Compare $\sin 2\theta: R\sin\alpha=\sqrt{3}$ (2)

Divide (2) by (1):

$\tan\alpha=\sqrt{3}\Rightarrow\alpha=\frac{\pi}{3}$

Square and add equations:

$R^2=1+3=4\Rightarrow R=2$

So $\cos 2\theta-\sqrt{3}\sin 2\theta\equiv 2\cos\left(2\theta+\frac{\pi}{3}\right)$

9 $4\cos\left(2\theta-\frac{\pi}{6}\right)\equiv 4\cos 2\theta\cos\frac{\pi}{6}+4\sin 2\theta\sin\frac{\pi}{6}$

$\equiv 2\sqrt{3}\cos 2\theta+2\sin 2\theta$

$\equiv 2\sqrt{3}\left(1-2\sin^2\theta\right)+4\sin\theta\cos\theta$

$\equiv 2\sqrt{3}-4\sqrt{3}\sin^2\theta+4\sin\theta\cos\theta$

10 a RHS $\equiv\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$

$\equiv\sqrt{2}\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)$

$\equiv\sqrt{2}\left(\sin\theta\times\frac{1}{\sqrt{2}}+\cos\theta\times\frac{1}{\sqrt{2}}\right)$

$\equiv\sin\theta+\cos\theta$

$\equiv$ LHS

b RHS $\equiv 2\sin\left(2\theta-\frac{\pi}{6}\right)$

$\equiv 2\left(\sin 2\theta\cos\frac{\pi}{6}-\cos 2\theta\sin\frac{\pi}{6}\right)$

$\equiv 2\left(\sin 2\theta\times\frac{\sqrt{3}}{2}-\cos 2\theta\times\frac{1}{2}\right)$

$\equiv\sqrt{3}\sin 2\theta-\cos 2\theta$

$\equiv$ LHS

Challenge

1 a $\cos(A+B)-\cos(A-B)\equiv\cos A\cos B-\sin A\sin B-(\cos A\cos B+\sin A\sin B)$

$\equiv -2\sin A\sin B$

b Let $A+B=P$ and $A-B=Q$

Solving simultaneously gives

$$2A=P+Q$$
$$A=\frac{P+Q}{2}$$

and

$$2B=P-Q$$
$$B=\frac{P-Q}{2}$$

Substituting these into the identity from part a gives

$$\cos P-\cos Q\equiv-2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

c Rearranging the identity from part a to give $\sin A\sin B\equiv-\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$

$$\begin{aligned}3\sin x\sin 7x &\equiv -\frac{3}{2}\cos(x+7x)+\frac{3}{2}\cos(x-7x)\\ &\equiv -\frac{3}{2}\cos 8x+\frac{3}{2}\cos(-6x)\\ &\equiv -\frac{3}{2}\cos 8x+\frac{3}{2}\cos(6x) \quad (\text{as } \cos(-x)\equiv\cos x)\\ &\equiv -\frac{3}{2}(\cos 8x-\cos 6x)\end{aligned}$$

2 a $\sin(A+B)+\sin(A-B)\equiv\sin A\cos B+\cos A\sin B+(\sin A\cos B-\cos A\sin B)$

$\equiv 2\sin A\cos B$

Let $A+B=P$ and $A-B=Q$

Solving simultaneously gives

$$2A=P+Q$$
$$A=\frac{P+Q}{2}$$

and

$$2B=P-Q$$
$$B=\frac{P-Q}{2}$$

Substituting these into the equation for $\sin(A+B)+\sin(A-B)$ gives

$$\sin P+\sin Q\equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

2 b Let $\frac{11\pi}{24} = \frac{P+Q}{2}, \frac{5\pi}{24} = \frac{P-Q}{2}$

$$\frac{22\pi}{24} = P+Q, \frac{10\pi}{24} = P-Q$$

Solving simultaneously gives:

$$2P = \frac{32\pi}{24},\ P = \frac{16\pi}{24}$$

and

$$2Q = \frac{12\pi}{24},\ Q = \frac{6\pi}{24}$$

So $2\sin\frac{11\pi}{24}\cos\frac{5\pi}{24} = \sin\left(\frac{16\pi}{24}\right) + \sin\left(\frac{6\pi}{24}\right) = \sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{3}+\sqrt{2}}{2}$