Trigonometry and Modelling Mixed Exercise

1 a i $\sin 40° \cos 10° - \cos 40° \sin 10°$

$= \sin(40° - 10°) = \sin 30° = \frac{1}{2}$

ii $\frac{1}{\sqrt{2}} \cos 15° - \frac{1}{\sqrt{2}} \sin 15°$

$\cos 45° \cos 15° - \sin 45° \sin 15°$

$\cos(45° + 15°) = \cos 60° = \frac{1}{2}$

iii $\frac{1 - \tan 15°}{1 + \tan 15°} = \frac{\tan 45° - \tan 15°}{1 + \tan 45° \tan 15°}$

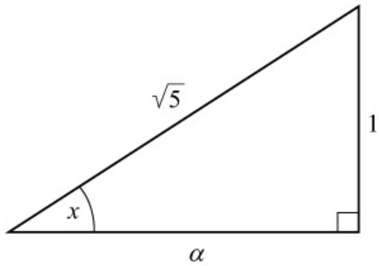
$= \tan(45° - 15°) = \tan 30° = \frac{\sqrt{3}}{3}$

2 As $\cos(x - y) = \sin y$

$\cos x \cos y + \sin x \sin y = \sin y \quad (1)$

Draw a right-angled triangle,

where $\sin x = \frac{1}{\sqrt{5}}$



Using Pythagoras' theorem,

$a^2 = \left(\sqrt{5}\right)^2 - 1 = 4 \quad \Rightarrow \quad a = 2$

So $\cos x = \frac{2}{\sqrt{5}}$

Substitute into (1):

$\frac{2}{\sqrt{5}} \cos y + \frac{1}{\sqrt{5}} \sin y = \sin y$

$\Rightarrow 2\cos y + \sin y = \sqrt{5} \sin y$

$\Rightarrow 2\cos y = \sin y\left(\sqrt{5} - 1\right)$

$\Rightarrow \frac{2}{\left(\sqrt{5} - 1\right)} = \tan y \qquad \left(\tan y = \frac{\sin y}{\cos y}\right)$

$\Rightarrow \tan y = \frac{2\left(\sqrt{5} + 1\right)}{\left(\sqrt{5} - 1\right)\left(\sqrt{5} + 1\right)}$

$= \frac{2\left(\sqrt{5} + 1\right)}{4} = \frac{\sqrt{5} + 1}{2}$

3 a $\tan A = 2$, $\tan B = \frac{1}{3}$ since $y = \frac{1}{3}x - \frac{1}{3}$

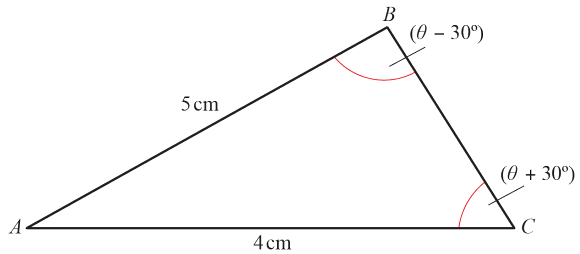
b The angle required is $(A - B)$

Using $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \frac{2 - \frac{1}{3}}{1 + 2 \times \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$

$\Rightarrow A - B = 45°$

4



Using $\frac{\sin B}{b}=\frac{\sin C}{c}$

$\Rightarrow \frac{\sin(\theta-30°)}{4}=\frac{\sin(\theta+30°)}{5}$

$\Rightarrow 5\sin(\theta-30°)=4\sin(\theta+30°)$

$\Rightarrow 5(\sin\theta\cos30°-\cos\theta\sin30°)$

$=4(\sin\theta\cos30°+\cos\theta\sin30°)$

$\Rightarrow \sin\theta\cos30°=9\cos\theta\sin30°$

$\Rightarrow \frac{\sin\theta}{\cos\theta}=9\frac{\sin30°}{\cos30°}=9\tan30°$

$\Rightarrow \tan\theta=9\times\frac{\sqrt{3}}{3}=3\sqrt{3}$

5 As the three values are consecutive terms of an arithmetic progression,

$\sin(\theta-30°)-\sqrt{3}\cos\theta=\sin\theta-\sin(\theta-30°)$

$\Rightarrow 2\sin(\theta-30°)=\sin\theta+\sqrt{3}\cos\theta$

$\Rightarrow 2(\sin\theta\cos30°-\cos\theta\sin30°)$

$=\sin\theta+\sqrt{3}\cos\theta$

$\Rightarrow \sqrt{3}\sin\theta-\cos\theta=\sin\theta+\sqrt{3}\cos\theta$

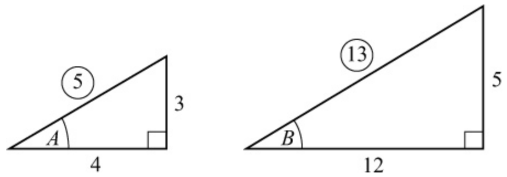
$\Rightarrow \sin\theta\left(\sqrt{3}-1\right)=\cos\theta\left(\sqrt{3}+1\right)$

$\Rightarrow \tan\theta=\frac{\sqrt{3}+1}{\sqrt{3}-1}$

Calculator value is $\theta=\tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1}=75°$

No other values as θ is acute.

6 a



$\sin A=\frac{3}{5}, \cos A=\frac{4}{5}$ $\quad \sin B=\frac{5}{13}, \cos B=\frac{12}{13}$

i $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$=\frac{3}{5}\times\frac{12}{13}+\frac{4}{5}\times\frac{5}{13}=\frac{56}{65}$

ii $\tan 2B=\frac{2\tan B}{1-\tan^2 B}$

$=\frac{2\times\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}=\frac{\frac{5}{6}}{\frac{119}{144}}$

$=\frac{5}{6}\times\frac{144}{119}=\frac{120}{119}$

b $\cos C=\cos\left(180°-(A+B)\right)$

$=-\cos(A+B)$

$=-(\cos A\cos B-\sin A\sin B)$

$=-\left(\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}\right)$

$=-\frac{33}{65}$

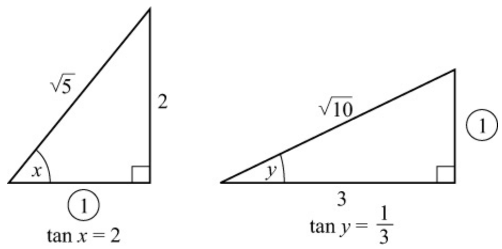
7 a $\cos 2x\equiv1-2\sin^2 x$

$=1-2\left(\frac{2}{\sqrt{5}}\right)^2=1-\frac{8}{5}=-\frac{3}{5}$

b $\cos 2y\equiv2\cos^2 y-1$

$=2\left(\frac{3}{\sqrt{10}}\right)^2-1=2\left(\frac{9}{10}\right)-1=\frac{4}{5}$

7 c



i $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \dfrac{2 + \frac{1}{3}}{1 - \frac{2}{3}} = \dfrac{\frac{7}{3}}{\frac{1}{3}} = 7$

ii $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y} = \dfrac{\frac{5}{3}}{\frac{5}{3}} = 1$

As x and y are acute, and $x > y$, $x - y$ is acute

So $x - y = \dfrac{\pi}{4}$ $\left(\text{it cannot be } \dfrac{5\pi}{4}\right)$

8 a $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$= \dfrac{1}{2} + \dfrac{1}{3} = \dfrac{5}{6}$

$5\sin(x-y) \equiv 5(\sin x \cos y - \cos x \sin y)$

$= 5\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = 5 \times \dfrac{1}{6} = \dfrac{5}{6}$

b $\dfrac{\sin x \cos y}{\cos x \sin y} = \dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{3}{2}$

$\Rightarrow \dfrac{\tan x}{\tan y} = \dfrac{3}{2}$

so $\tan x = \dfrac{3\tan y}{2} = \dfrac{3k}{2}$

c $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x} = \dfrac{3k}{1 - \frac{9}{4}k^2}$

$= \dfrac{12k}{4 - 9k^2}$

9 a $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$

$\sqrt{3}\sin 2\theta = 1 - 2\sin^2\theta = \cos 2\theta$

$\dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{1}{\sqrt{3}} \Rightarrow \tan 2\theta = \dfrac{1}{\sqrt{3}}$

b $\tan 2\theta = \dfrac{1}{\sqrt{3}}$, for $0 \le 2\theta \le 2\pi$

$2\theta = \dfrac{\pi}{6}, \dfrac{7\pi}{6} \Rightarrow \theta = \dfrac{\pi}{12}, \dfrac{7\pi}{12}$

10 a $\cos 2\theta = 5\sin\theta$

$\Rightarrow \cos 2\theta - 5\sin\theta = 0$

$\Rightarrow 1 - 2\sin^2\theta - 5\sin\theta = 0$

$\Rightarrow 2\sin^2\theta + 5\sin\theta - 1 = 0$

$a = 2$, $b = 5$ and $c = -1$

b $2\sin^2\theta + 5\sin\theta - 1 = 0$

Using the quadratic formula

$\sin\theta = \dfrac{-5 \pm \sqrt{5^2 - 4(2)(-1)}}{2(2)}$

$= \dfrac{-5 \pm \sqrt{33}}{4}$

$\sin\theta = 0.1861$, for $-\pi \le \theta \le \pi$

$\sin\theta$ is positive so solutions in the first and second quadrants

$\theta = \sin^{-1} 0.1861, \pi - \sin^{-1} 0.1861$

$\theta = 0.187, 2.954$ (3 d.p.)

11 a $\cos(x - 60°) = \cos x \cos 60° + \sin x \sin 60°$

$= \dfrac{1}{2}\cos x + \dfrac{\sqrt{3}}{2}\sin x$

So $2\sin x = \dfrac{1}{2}\cos x + \dfrac{\sqrt{3}}{2}\sin x$

$\Rightarrow \left(2 - \dfrac{\sqrt{3}}{2}\right)\sin x = \dfrac{1}{2}\cos x$

$\Rightarrow \tan x = \dfrac{\frac{1}{2}}{2 - \frac{\sqrt{3}}{2}} = \dfrac{1}{2} \times \dfrac{2}{4 - \sqrt{3}} = \dfrac{1}{4 - \sqrt{3}}$

b $\tan x = \dfrac{1}{4 - \sqrt{3}} = 0.44$ (2 d.p.), in the interval $0° \le \theta \le 360°$

$\tan\theta$ is positive so solutions in the first and third quadrants

$x = 23.8°, 203.8°$ (1 d.p.)

12 a $\cos(x+20°) = \sin(90° - 20° - x)$
$= \sin(70° - x)$
$= \sin 70° \cos x - \cos 70° \sin x \quad (1)$

$4\sin(70° + x) = 4\sin 70° \cos x + 4\cos 70° \sin x \quad (2)$

As (1) = (2)

$4\sin 70° \cos x + 4\cos 70° \sin x = \sin 70° \cos x - \cos 70° \sin x$

$5\sin x \cos 70° = -3\sin 70° \cos x$

$\tan x = -\frac{3}{5}\tan 70°$

b $\tan x = -\frac{3}{5}\tan 70°$, for $0° \le \theta \le 180°$

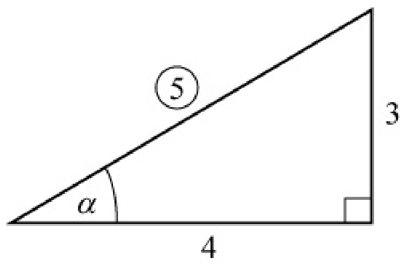
$\tan\theta$ is negative so the solution is in the second quadrant

$x = 180° + \tan^{-1}\left(-\frac{3}{5}\tan 70°\right)$

$x = 180° - \tan^{-1}(-1.648)$

$x = 180° - (-58.8°) = 121.2°$ (1 d.p.)

13 a Draw a right-angled triangle and find $\sin\alpha$ and $\cos\alpha$.



$\Rightarrow \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = \frac{4}{5}$

$3\sin(\theta+\alpha) + 4\cos(\theta+\alpha)$
$\equiv 3(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + 4(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$
$\equiv 3\left(\frac{4}{5}\sin\theta + \frac{3}{5}\cos\theta\right) + 4\left(\frac{4}{5}\cos\theta - \frac{3}{5}\sin\theta\right)$
$\equiv \frac{12}{5}\sin\theta + \frac{9}{5}\cos\theta + \frac{16}{5}\cos\theta - \frac{12}{5}\sin\theta$
$\equiv \frac{25}{5}\cos\theta \equiv 5\cos\theta$

b $\cos(x+270°)$
$\equiv \cos x° \cos 270° - \sin x° \sin 270°$
$= (-0.8)(0) - (0.6)(-1)$
$= 0 + 0.6 = 0.6$

$\cos(x+540°)$
$\equiv \cos x° \cos 540° - \sin x° \sin 540°$
$= (-0.8)(-1) - (0.6)(0)$
$= 0.8 - 0 = 0.8$

14 a One example is sufficient to disprove a statement. Let $A = 60°,\ B = 0°$

$\sec(A+B) = \sec(60° + 0°)$
$= \sec 60° = \frac{1}{\cos 60°} = 2$

$\sec A = \sec 60° = \frac{1}{\cos 60°} = 2$

$\sec B = \sec 0° = \frac{1}{\cos 0°} = 1$

So $\sec A + \sec B = 2 + 1 = 3$

So $\sec(60° + 0°) \ne \sec 60° + \sec 0°$

$\Rightarrow \quad \sin(A+B) \equiv \sec A + \sec B$ is not true for all values of $A,\ B$.

b LHS $\equiv \tan\theta + \cot\theta$
$\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$
$\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$
$\equiv \frac{1}{\frac{1}{2}\sin 2\theta}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$, and $\sin 2\theta \equiv 2\sin\theta\cos\theta$

So LHS $\equiv \frac{2}{\sin 2\theta}$
$\equiv 2\operatorname{cosec} 2\theta$
$\equiv$ RHS

15 a Using $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ with $\theta = \dfrac{\pi}{8}$

$\Rightarrow \quad \tan\dfrac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $t = \tan\dfrac{\pi}{8}$

So $1 = \dfrac{2t}{1-t^2}$

$\Rightarrow \quad 1-t^2 = 2t$

$\Rightarrow \quad t^2+2t-1=0$

$\Rightarrow \quad t = \dfrac{-2\pm\sqrt{8}}{2} = \dfrac{-2\pm2\sqrt{2}}{2}$

$= -1\pm\sqrt{2}$

As $\dfrac{\pi}{8}$ is acute, $\tan\dfrac{\pi}{8}$ is positive, so $\tan\dfrac{\pi}{8} = \sqrt{2}-1$

b $\tan\dfrac{3\pi}{8} = \tan\left(\dfrac{\pi}{4}+\dfrac{\pi}{8}\right) = \dfrac{\tan\frac{\pi}{4}+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{8}}$

$= \dfrac{1+\left(\sqrt{2}-1\right)}{1-\left(\sqrt{2}-1\right)} = \dfrac{\sqrt{2}}{2-\sqrt{2}}$

$= \dfrac{\sqrt{2}\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)\left(2+\sqrt{2}\right)}$

$= \dfrac{\sqrt{2}}{2}\left(2+\sqrt{2}\right) = \sqrt{2}+1$

16 a Let $\sin x - \sqrt{3}\cos x \equiv R\sin(x-\alpha)$

$\equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

$R>0,\ 0<\alpha<90°$

Compare $\sin x$: $R\cos\alpha = 1$ (1)

Compare $\cos x$: $R\sin\alpha = \sqrt{3}$ (2)

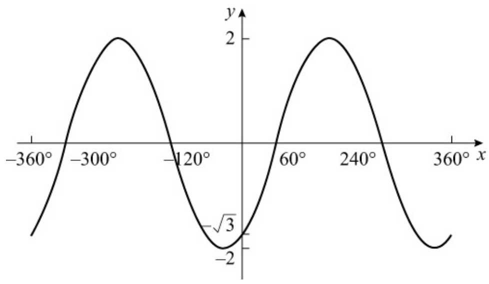
Divide (2) by (1): $\tan\alpha = \sqrt{3}$

$\Rightarrow \quad \alpha = 60°$

$R^2 = \left(\sqrt{3}\right)^2 + 1^2 = 4 \ \Rightarrow \ R = 2$

So $\sin x - \sqrt{3}\cos x \equiv 2\sin(x-60°)$

b Sketch $y = 2\sin(x-60°)$ by first translating $y = \sin x$ by $60°$ to the right and then stretching the result in the y direction by scale factor 2.



Graph meets y-axis when $x = 0$,

i.e. $y = 2\sin(-60°) = -\sqrt{3}$, at $\left(0,-\sqrt{3}\right)$

Graph meets x-axis when $y = 0$,

i.e. $(-300°,\ 0)$, $(-120°,\ 0)$, $(60°,\ 0)$, $240°,\ 0)$

17 a Let $7\cos2\theta + 24\sin2\theta \equiv R\cos(2\theta-\alpha)$

$\equiv R\cos2\theta\cos\alpha + R\sin2\theta\sin\alpha$

$R>0,\ 0<\alpha<\dfrac{\pi}{2}$

Compare $\cos2\theta$: $R\cos\alpha = 7$ (1)

Compare $\sin2\theta$: $R\sin\alpha = 24$ (2)

Divide (2) by (1): $\tan\alpha = \dfrac{24}{7}$

$\Rightarrow \quad \alpha = 1.29$ (2 d.p.)

$R^2 = 24^2+7^2 \ \Rightarrow \ R = 25$

So $7\cos2\theta + 24\sin2\theta \equiv 25\cos(2\theta-1.29)$

b $14\cos2\theta + 48\sin\theta\cos\theta$

$\equiv 14\left(\dfrac{1+\cos2\theta}{2}\right) + 24(2\sin\theta\cos\theta)$

$\equiv 7(1+\cos2\theta) + 24\sin2\theta$

$\equiv 7 + 7\cos2\theta + 24\sin2\theta$

The maximum value of $7\cos2\theta + 24\sin2\theta$ is 25

(using (a) with $\cos(2\theta-1.29) = 1$)

So maximum value of

$7 + 7\cos2\theta + 24\sin2\theta = 7 + 25 = 32$

17 c Using the answer to part a:
Solve $25\cos(2\theta - 1.29) = 12.5$

$\cos(2\theta - 1.29) = \frac{1}{2}$

$2\theta - 1.29 = -\frac{\pi}{3}, \frac{\pi}{3}$

$\theta = 0.119902..., 1.167099...$

$\theta = 0.12, 1.17$ (2 d.p.)

18 a Let $1.5\sin 2x + 2\cos 2x \equiv R\sin(2x + \alpha)$
$\equiv R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$

$R > 0, 0 < \alpha < \frac{\pi}{2}$

Compare $\sin 2x : R\cos\alpha = 1.5$ (1)

Compare $\cos 2x : R\sin\alpha = 2$ (2)

Divide (2) by (1) : $\tan\alpha = \frac{4}{3}$

$\Rightarrow \alpha = 0.927$ (3 d.p.)

$R^2 = 2^2 + 1.5^2 \Rightarrow R = 2.5$

b $3\sin x\cos x + 4\cos^2 x$

$\equiv \frac{3}{2}(2\sin x\cos x) + 4\left(\frac{1+\cos 2x}{2}\right)$

$\equiv \frac{3}{2}\sin 2x + 2 + 2\cos 2x$

$\equiv \frac{3}{2}\sin 2x + 2\cos 2x + 2$

c From part (a) $1.5\sin 2x + 2\cos 2x$
$\equiv 2.5\sin(2x + 0.927)$

So maximum value of

$1.5\sin 2x + 2\cos 2x = 2.5 \times 1 = 2.5$

So maximum value of

$3\sin x\cos x + 4\cos^2 x = 2.5 + 2 = 4.5$

19 a $\sin^2\frac{\theta}{2} = 2\sin\theta$

$\frac{1-\cos\theta}{2} = 2\sin\theta$

$1 - \cos\theta = 4\sin\theta$

$4\sin\theta + \cos\theta = 1$

Let $4\sin\theta + \cos\theta = R\sin(\theta + \alpha)$
$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

So $R\cos\alpha = 4$ and $R\sin\alpha = 1$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{4}$

$\alpha = \tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}0.25 = 14.04$ (2 d.p.)

$R^2 = 4^2 + 1^2 = \sqrt{17}$

$4\sin\theta + \cos\theta = \sqrt{17}\sin(\theta + 14.04°) = 1$

b $\sqrt{17}\sin(\theta + 14.04°) = 1$, for $0° \leq \theta \leq 360°$

$\sin(\theta + 14.04°) = \frac{1}{\sqrt{17}} = 0.24$ (2 d.p.)

$\theta + 14.04° = \sin^{-1}0.24 = 14.04°$, for
$14.04° \leq \theta + 14.04° \leq 374.04°$

$\theta + 14.04° = 14.04°, 165.96°, 374.04°$

$\theta = 0°, 151.9°, 360°$

20 a $2\cos\theta = 1 + 3\sin\theta$
So $2\cos\theta - 3\sin\theta = 1$
Let $2\cos\theta - 3\sin\theta = R\cos(\theta + \alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = 3$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{3}{2}$

$\alpha = \tan^{-1}\left(\frac{3}{2}\right) = 56.3°$ (1 d.p.)

$R^2 = 2^2 + 3^2 = 13$

$R = \sqrt{13}$

So

$2\cos\theta - 3\sin\theta = \sqrt{13}\cos(\theta + 56.3°) = 1$

20 b $\sqrt{13}\cos(\theta+56.3°)=1$, for $0°\le\theta\le 360°$

$\cos(\theta+56.3°)=\dfrac{1}{\sqrt{13}}$,

for $56.3°\le\theta+56.3°\le 416.3°$

$\theta+56.3°=73.9°, 286.1°$ (1 d.p.)

$\theta=17.6°, 229.8°$ (1 d.p.)

21 a LHS $\equiv\dfrac{1}{\cos\theta}\times\dfrac{1}{\sin\theta}\equiv\dfrac{1}{\frac{1}{2}\sin 2\theta}$

$\equiv\dfrac{2}{\sin 2\theta}\equiv 2\operatorname{cosec}2\theta\equiv$ RHS

b LHS $\equiv\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}-\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}$

$\equiv\dfrac{1+\tan x}{1-\tan x}-\dfrac{1-\tan x}{1+\tan x}$

$\equiv\dfrac{(1+\tan x)^2-(1-\tan x)^2}{(1-\tan x)(1+\tan x)}$

$\equiv\dfrac{(1+2\tan x+\tan^2 x)}{1-\tan^2 x}-\dfrac{(1-2\tan x+\tan^2 x)}{1-\tan^2 x}$

$\equiv\dfrac{4\tan x}{1-\tan^2 x}$

$\equiv 2\left(\dfrac{2\tan x}{1-\tan^2 x}\right)$

$\equiv 2\tan 2x\equiv$ RHS

c LHS $\equiv(\sin x\cos y+\cos x\sin y)$

$\times(\sin x\cos y-\cos x\sin y)$

$\equiv\sin^2 x\cos^2 y-\cos^2 x\sin^2 y$

$\equiv(1-\cos^2 x)\cos^2 y$

$-\cos^2 x(1-\cos^2 y)$

$\equiv\cos^2 y-\cos^2 x\cos^2 y$

$-\cos^2 x+\cos^2 x\cos^2 y$

$\equiv\cos^2 y-\cos^2 x=$ RHS

d LHS $\equiv 1+2\cos 2\theta+(2\cos^2 2\theta-1)$

$\equiv 2\cos 2\theta+2\cos^2 2\theta$

$\equiv 2\cos 2\theta(1+\cos 2\theta)$

$\equiv 2\cos 2\theta(2\cos^2\theta)$

$\equiv 4\cos^2\theta\cos 2\theta=$ RHS

22 a LHS $\equiv\dfrac{1-\cos 2x}{1+\cos 2x}\equiv\dfrac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}$

$\equiv\dfrac{2\sin^2 x}{2\cos^2 x}\equiv\tan^2 x=$ RHS

b $\tan^2 x=3$

$\tan x=\pm\sqrt{3}$, for $-\pi\le x\le\pi$

$\tan x=\sqrt{3}\Rightarrow x=\dfrac{\pi}{3}, -\dfrac{2\pi}{3}$

$\tan x=-\sqrt{3}\Rightarrow x=-\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$x=-\dfrac{2\pi}{3}, -\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

23 a LHS $\equiv\cos^4 2\theta-\sin^4 2\theta$

$\equiv(\cos^2 2\theta-\sin^2 2\theta)(\cos^2 2\theta+\sin^2 2\theta)$

$\equiv(\cos^2 2\theta-\sin^2 2\theta)(1)$

$\equiv\cos 4\theta\equiv$ RHS

b $\cos 4\theta=\dfrac{1}{2}$, for $0°\le 4\theta\le 720°$

$4\theta=60°, 300°, 420°, 660°$

$\theta=15°, 75°, 105°, 165°$

24 a LHS $\equiv\dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$

$\equiv\dfrac{2\sin^2\theta}{2\sin\theta\cos\theta}$

$\equiv\dfrac{\sin\theta}{\cos\theta}\equiv\tan\theta=$ RHS

b When $\theta=180°$, $\sin 2\theta=\sin 360°=0$ and $2-2\cos 360°=2-2=0$ therefore $\theta=180°$ is a solution of the equation $\sin 2\theta=2-2\cos 2\theta$

24 c Rearrange $\sin 2\theta = 2 - 2\cos 2\theta$ to give

$$\frac{2(1-\cos 2\theta)}{\sin 2\theta} = 1$$

Using the identity in part (a) gives

$2\tan\theta = 1$

$\Rightarrow \tan\theta = \frac{1}{2}$, for $0 < \theta < 360°$

$\theta = 26.6°, 206.6°$ (1 d.p.)

25 a Set $2\cos x - \sqrt{5}\sin x \equiv R\cos(x+\alpha)$

$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = \sqrt{5}$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{\sqrt{5}}{2}$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) = 0.841 \text{ (3 d.p.)}$$

$R^2 = 2^2 + \left(\sqrt{5}\right)^2 = 9$

$R = 3$

$2\cos x - \sqrt{5}\sin x \equiv 3\cos(x + 0.841)$

b $3\cos(x+0.841) = -1$,

for $0.841 \le x + 0.841 < 2\pi + 0.841$

$$\cos(x+0.841) = -\frac{1}{3}$$

$x + 0.841 = 1.911, 4.372$

$x = 1.07, 3.53$ (2 d.p.)

26 a Set $1.4\sin\theta - 5.6\cos\theta \equiv R\sin(\theta - \alpha)$

$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

So $R\cos\alpha = 1.4$ and $R\sin\alpha = 5.6$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{5.6}{1.4}$$

$\alpha = \tan^{-1} 4 == 75.964°$ (3 d.p.)

$R^2 = 1.4^2 + 5.6^2 = 33.32$

$R = 5.772$ (3 d.p.)

b The maximum value of $5.772\sin(\theta - 75.964)°$ is when $\sin(\theta - 75.964)° = 1$. So the maximum value is 5.772 and it occurs when $\theta - 75.964° = 90°$, $\theta = 165.964°$

c $$12 - 5.6\cos\left(\frac{360t}{365}\right)° + 1.4\sin\left(\frac{360t}{365}\right)°$$

$$\equiv 12 + 5.772\sin\left(\frac{360t}{365} - 75.964\right)°$$

The minimum number of daylight hours is when $\sin\left(\frac{360t}{365} - 75.964\right)° = -1$

So minimum is $12 - 5.772 = 6.228$ hours

d $$\sin\left(\frac{360t}{365} - 75.964\right)° = -1$$

$$\frac{360t}{365} - 75.964 = 270°$$

$t = 351$ days

27 a Let $12\sin x + 5\cos x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{5}{12}$$

$$\alpha = \tan^{-1}\left(\frac{5}{12}\right) = 22.6° \text{ (1 d.p.)}$$

$R^2 = 12^2 + 5^2 = 169$

$R = 13$

So $12\sin x + 5\cos x = 13\sin(x + 22.6°)$

b $$\text{v}(x) = \frac{50}{12\sin\left(\frac{2x}{5}\right)° + 5\cos\left(\frac{2x}{5}\right)°}$$

$$= \frac{50}{13\sin\left(\frac{2x}{5} + 22.6°\right)}$$

The minimum value of v is when

$$\sin\left(\frac{2x}{5} + 22.6\right)° = 1$$

So $\frac{50}{13} = 3.85$ m/s (2 d.p.)

27 c $\sin\left(\frac{2x}{5}+22.6°\right)=1$, for

$$22.6° \le \frac{2x}{5}+22.6° \le 166.6°$$

$$\frac{2x}{5}+22.6°=90°$$

$x=168.5$ minutes

Challenge

1 a Write $\cos 2\theta$ as $\cos(3\theta-\theta)$ and write $\cos 4\theta$ as $\cos(3\theta+\theta)$.

Then, using $\cos(A \pm B) = \cos A \cos B \mp \sin A \cos B$,

$$\cos 2\theta = \cos 3\theta \cos\theta + \sin 3\theta \sin\theta$$

$$\cos 4\theta = \cos 3\theta \cos\theta - \sin 3\theta \sin\theta$$

$$\Rightarrow \quad \cos 2\theta + \cos 4\theta = 2\cos 3\theta \cos\theta$$

Similarly, using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$,

$$\sin 2\theta = \sin 3\theta \cos\theta - \cos 3\theta \sin\theta$$

$$\sin 4\theta = \sin 3\theta \cos\theta + \cos 3\theta \sin\theta$$

$$\Rightarrow \sin 2\theta - \sin 4\theta = -2\cos 3\theta \sin\theta$$

Therefore,

$$\frac{\cos 2\theta+\cos 4\theta}{\sin 2\theta-\sin 4\theta}=\frac{2\cos 3\theta\cos\theta}{-2\cos 3\theta\sin\theta}=-\frac{\cos\theta}{\sin\theta}=-\cot\theta \quad \text{as required.}$$

$$\equiv\frac{2\cos\left(\frac{6\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)}{2\cos\left(\frac{6\theta}{2}\right)\sin\left(\frac{-2\theta}{2}\right)}\equiv\frac{2\cos 3\theta\cos\theta}{2\cos 3\theta\sin(-\theta)}\equiv\frac{\cos\theta}{\sin(-\theta)}\equiv-\cot\theta$$

b LHS $\equiv \cos x + 2\cos 3x + \cos 5x$

$$\equiv \cos 5x + \cos x + 2\cos 3x$$

$$\equiv 2\cos\left(\frac{6x}{2}\right)\cos\left(\frac{4x}{2}\right)+2\cos 3x$$

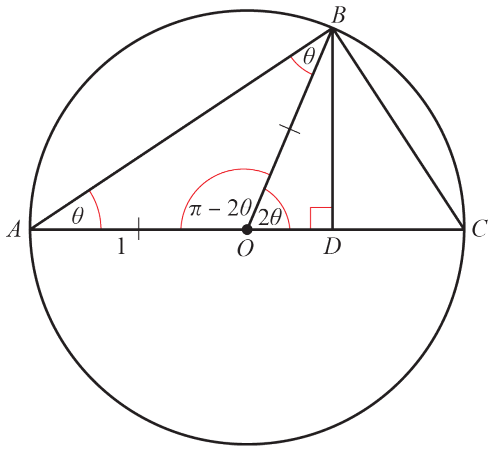
$$\equiv 2\cos 3x\cos 2x + 2\cos 3x$$

$$\equiv 2\cos 3x(\cos 2x+1)$$

$$\equiv 2\cos 3x(2\cos^2 x)$$

$$\equiv 4\cos^2 x\cos 3x \equiv \text{RHS}$$

2 a As $\angle OAB = \angle OBA \Rightarrow \angle AOB = \pi - 2\theta$, so $\angle BOD = 2\theta$



$OB = 1$

$OD = \cos 2\theta$

$BD = \sin 2\theta$

$AB = 2\cos\theta$

$$\sin\theta = \frac{BD}{AB} = \frac{BD}{2\cos\theta}$$

So $BD = 2\sin\theta\cos\theta$

But $BD = \sin 2\theta$

So $\sin 2\theta \equiv 2\sin\theta\cos\theta$

b $AB = 2\cos\theta$

$AD = (2\cos\theta)\cos\theta = 2\cos^2\theta$

$OD = 2\cos^2\theta - 1$

From part **a**, $OD = \cos 2\theta$

So $\cos 2\theta \equiv 2\cos^2\theta - 1$