Parametric equations 8B

1 a $x = 2\sin t - 1$

So $\sin t = \dfrac{x+1}{2}$ (1)

$y = 5\cos t + 4$

$\cos t = \dfrac{y-4}{5}$ (2)

Substitute (1) and (2) into $\sin^2 t + \cos^2 t \equiv 1$:

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-4}{5}\right)^2 = 1$$

$$\frac{(x+1)^2}{4} + \frac{(y-4)^2}{25} = 1$$

$$25(x+1)^2 + 4(y-4)^2 = 100$$

b $y = \sin 2t$

$= 2\sin t\cos t$

So, since $x = \cos t$,

$y = 2x\sin t$ (1)

$\sin^2 t + \cos^2 t \equiv 1$

$\sin^2 t \equiv 1 - \cos^2 t = 1 - x^2$

$\sin t = \sqrt{1-x^2}$ (2)

Substitute (2) into (1):

$y = 2x\sqrt{1-x^2}$

or $y^2 = 4x^2(1-x^2)$

c $y = 2\cos 2t$

$= 2(2\cos^2 t - 1)$

So, since $x = \cos t$,

$y = 2(2x^2 - 1)$

$y = 4x^2 - 2$

d $y = \tan 2t$

So $y = \dfrac{2\tan t}{1-\tan^2 t}$ (1)

$\sin^2 t + \cos^2 t \equiv 1$

$\cos^2 t \equiv 1 - \sin^2 t = 1 - x^2$

$\cos t = \sqrt{1-x^2}$ (2)

Substitute (2) and $x = \sin t$ into (1):

$$y = \frac{2\dfrac{\sin t}{\cos t}}{1-\dfrac{\sin^2 t}{\cos^2 t}} = \frac{\dfrac{2x}{\sqrt{1-x^2}}}{1-\dfrac{x^2}{1-x^2}} = \frac{\dfrac{2x}{\sqrt{1-x^2}}}{\dfrac{1-2x^2}{1-x^2}}$$

$$= \frac{2x(1-x^2)}{(1-2x^2)\sqrt{1-x^2}}$$

Hence $y = \dfrac{2x\sqrt{1-x^2}}{1-2x^2}$

e $x = \cos t + 2$

$\cos t = x - 2$ (1)

$y = \sec t = \dfrac{4}{\cos t}$

$\cos t = \dfrac{4}{y}$ (2)

Substitute (1) into (2):

$x - 2 = \dfrac{4}{y}$

$y = \dfrac{4}{x-2}$

f $x = 3\cot t$

$\cot t = \dfrac{x}{3}$ (1)

$\text{cosec}\, t = y$ (2)

Substitute (1) and (2) into $1 + \cot^2 t \equiv \text{cosec}^2 t$:

$$1 + \left(\frac{x}{3}\right)^2 = y^2$$

$$y^2 = 1 + \frac{x^2}{9}$$

2 a $x = \sin t - 5$

$\Rightarrow \sin t = x + 5 \quad (1)$

$y = \cos t + 2$

$\Rightarrow \cos t = y - 2 \quad (2)$

Substitute (1) and (2) into $\sin^2 t + \cos^2 t \equiv 1$:

$(x+5)^2 + (y-2)^2 = 1$

b This is a circle with centre (−5, 2) and radius 1

c One full revolution around the circle is obtained for an interval of t corresponding to one period of both parametric equations $y = \cos t + 2$ and $x = \sin t - 5$.

So $0 \le t \le 2\pi$ is a suitable domain.

3 $x = 4\sin t + 3$

$4\sin t = x - 3$

$\therefore \sin t = \dfrac{x-3}{4} \quad (1)$

$y = 4\cos t - 1$

$4\cos t = y + 1$

$\therefore \cos t = \dfrac{y+1}{4} \quad (2)$

Substitute (1) and (2) into $\sin^2 t + \cos^2 t = 1$:

$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{4}\right)^2 = 1$$

$$\frac{(x-3)^2}{4^2} + \frac{(y+1)^2}{4^2} = 1$$

$$\frac{(x-3)^2}{16} + \frac{(y+1)^2}{16} = 1$$

$(x-3)^2 + (y+1)^2 = 16$

So the radius of the circle is 4 and the centre is (3, −1).

4 $x = \cos t - 2$

$\Rightarrow \cos t = x + 2 \quad (1)$

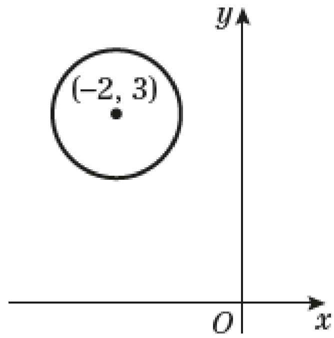
$y = \sin t + 3$

$\Rightarrow \sin t = y - 3 \quad (2)$

Substitute (1) and (2) into $\sin^2 t + \cos^2 t \equiv 1$:

$(x+2)^2 + (y-3)^2 = 1$

This is a circle with centre (−2, 3) and radius 1:



5 a $y = \sin\left(t + \dfrac{\pi}{4}\right)$

$$= \sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t$$

$$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\cos t \quad (1)$$

(since $x = \sin t$)

$\sin^2 t + \cos^2 t \equiv 1$

$\cos^2 t \equiv 1 - \sin^2 t = 1 - x^2$

$\therefore \cos t = \sqrt{1-x^2} \quad (2)$

Substitute (2) into (1):

$$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\sqrt{1-x^2}$$

or $y = \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2(1-x^2)}}{2}$

$x = \sin t, \ -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

$\Rightarrow \ -1 < x < 1$

5 b $x = 3\cos t$

$\Rightarrow \cos t = \dfrac{x}{3}$

$y = 2\cos\left(t + \dfrac{\pi}{6}\right)$

$= 2\cos t \cos\dfrac{\pi}{6} - 2\sin t \sin\dfrac{\pi}{6}$

$= 2\cos t \times \dfrac{\sqrt{3}}{2} - 2\sin t \times \dfrac{1}{2}$

$= \sqrt{3}\cos t - \sin t$

So $y = \dfrac{\sqrt{3}}{3}x - \sin t$ (1)

$\sin^2 t + \cos^2 t \equiv 1$

$\sin^2 t \equiv 1 - \cos^2 t = 1 - \left(\dfrac{x}{3}\right)^2$

$\sin t = \sqrt{1 - \left(\dfrac{x}{3}\right)^2}$ (2)

Substitute (2) into (1):

$y = \dfrac{\sqrt{3}}{3}x - \sqrt{1 - \left(\dfrac{x}{3}\right)^2}$

$= \dfrac{\sqrt{3}}{3}x - \sqrt{\dfrac{9 - x^2}{9}}$

$\therefore\ y = \dfrac{\sqrt{3}}{3}x - \dfrac{\sqrt{9 - x^2}}{3}$

$x = 3\cos t,\quad 0 < t < \dfrac{\pi}{3}$

$\Rightarrow\ \dfrac{3}{2} < x < 3$

c $y = 3\sin(t + \pi)$

$= 3\sin t \cos\pi + 3\cos t \sin\pi$

$= 3\sin t \times (-1) + 3\cos t \times 0$

$= -3\sin t$

Since $x = \sin t$,

$y = -3x$

$x = \sin t,\quad 0 < t < 2\pi$

$\Rightarrow -1 < x < 1$

6 a $x = 8\cos t$

$\cos t = \dfrac{x}{8}$

So $y = \dfrac{1}{4}\sec^2 t = \dfrac{1}{4\cos^2 t}$

$= \dfrac{1}{4\left(\dfrac{x}{8}\right)^2} = \dfrac{1}{4} \times \dfrac{64}{x^2} = \dfrac{16}{x^2}$

Therefore a Cartesian equation for C is

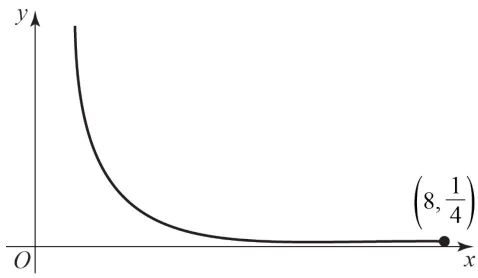
$y = \dfrac{16}{x^2}$

$x = 8\cos t,\quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

$\Rightarrow\ 0 < x < 8$

b For $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$ the range of the parametric equation $x = 8\cos t$ is $0 \le x \le 8$, so the domain of $y = f(x)$ is $0 \le x \le 8$.

The range of the parametric equation $y = \dfrac{1}{4}\sec^2 t$ is $y \ge \dfrac{1}{4}$, so the range of $y = f(x)$ is $y \ge \dfrac{1}{4}$



7 $x = 3\cot^2 2t$

$\cot^2 2t = \frac{x}{3}$

$\frac{x}{3} = \frac{\cos^2 2t}{\sin^2 2t} = \frac{1-\sin^2 2t}{\sin^2 2t} = \frac{1}{\sin^2 2t} - 1$

$\frac{x}{3} + 1 = \frac{1}{\sin^2 2t}$

$\frac{x+3}{3} = \frac{1}{\sin^2 2t}$

$\sin^2 2t = \frac{3}{x+3}$

$\therefore\ y = 3\sin^2 2t = 3 \times \frac{3}{x+3} = \frac{9}{x+3}$

For $0 < t \leq \frac{\pi}{4}$ the range of the parametric function $x = 3\cot^2 2t$ is $x \geq 0$, so the domain of f(x) is $x \geq 0$.

8 a $x = \frac{1}{3}\sin t$

$\Rightarrow \sin t = 3x$

$y = \sin 3t = \sin(t + 2t)$

$= \sin t \cos 2t + \cos t \sin 2t$

$= \sin t(1 - 2\sin^2 t) + \cos t(2\sin t \cos t)$

$= \sin t(1 - 2\sin^2 t) + 2\sin t(1 - \sin^2 t)$

$= 3x(1 - 2 \times 9x^2) + 6x(1 - 9x^2)$

$= 3x - 54x^3 + 6x - 54x^3$

$= 9x - 108x^3$

$= 9x(1 - 12x^2)$

So the Cartesian equation of the curve is $y = 9x(1 - 12x^2)$, which is in the form $y = ax(1 - bx^2)$ with $a = 9$ and $b = 12$.

b For $0 < t < \frac{\pi}{2}$ the range of the parametric function $x = \frac{1}{3}\sin t$ is $0 < x < \frac{1}{3}$

so the domain of y = f(x) is $0 < x < \frac{1}{3}$

For $0 < t < \frac{\pi}{2}$ the range of the parametric function $y = \sin 3t$ is $-1 < y < 1$

so the range of y = f(x) is $-1 < y < 1$.

9 $x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$

$y = \sin\left(t - \frac{\pi}{6}\right)$

$= \sin t \cos\frac{\pi}{6} - \cos t \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t$

$\therefore\ y = \frac{\sqrt{3}}{2}\sin t - \frac{1}{4}x \qquad (1)$

$\sin^2 t + \cos^2 t \equiv 1$

$\sin^2 t \equiv 1 - \cos^2 t = 1 - \frac{x^2}{4}$

$\therefore\ \sin t = \sqrt{1 - \frac{x^2}{4}} \qquad (2)$

Substitute (2) into (1):

$y = \frac{\sqrt{3}}{2}\sqrt{1 - \frac{x^2}{4}} - \frac{1}{4}x$

$= \frac{1}{2}\sqrt{\frac{12 - 3x^2}{4}} - \frac{1}{4}x$

So the Cartesian equation is

$y = \frac{1}{4}\left(\sqrt{12 - 3x^2} - x\right)$

For $0 < t < \pi$, the range of the parametric function $x = 2\cos t$ is $-2 < x < 2$,
so the domain of y = f(x) is $-2 < x < 2$.

10 a $y = 5\sin t$

So $\sin t = \frac{y}{5}$

$\sin^2 t = \frac{y^2}{25}$

$x = \tan^2 t + 5$

$\tan^2 t = x - 5$

$\frac{\sin^2 t}{\cos^2 t} = x - 5$

$\frac{\sin^2 t}{1 - \sin^2 t} = x - 5$

$\frac{1}{x-5} = \frac{1}{\sin^2 t} - 1$

$\frac{1}{x-5} + 1 = \frac{1}{\frac{y^2}{25}}$

$\frac{x-4}{x-5} = \frac{25}{y^2}$

$\therefore\ y^2 = 25\left(\frac{x-5}{x-4}\right) = 25\left(1 - \frac{1}{x-4}\right)$

b For $0 < t < \frac{\pi}{2}$, the range of the parametric function $x = \tan^2 t + 5$ is $x > 5$,
so the domain of the curve is $x > 5$.
The range of the parametric function $y = 5\sin t$ is $0 < y < 5$,
so the range of the curve is $0 < y < 5$.

11 $y = 3\sin(t - \pi)$

$= 3\sin t\cos\pi - 3\cos t\sin\pi$

$= 3\sin t \times (-1) - 3\cos t \times 0$

$= -3\sin t$

So $\sin t = -\frac{y}{3}$

$\sin^2 t + \cos^2 t \equiv 1$

$\cos^2 t \equiv 1 - \sin^2 t$

$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \frac{y^2}{9}}$

$\therefore\ x = \tan t = \frac{\sin t}{\cos t} = \frac{-\frac{y}{3}}{\sqrt{1 - \frac{y^2}{9}}} = \frac{-y}{\sqrt{9 - y^2}}$

Therefore $x = -\frac{y}{\sqrt{9 - y^2}}$ is a Cartesian equation for C.

Challenge

$$x = \frac{1}{2}\cos 2t$$
$$= \frac{1}{2}(2\cos^2 t - 1)$$
$$\therefore \frac{2x+1}{2} = \cos^2 t$$
$$\cos t = \sqrt{\frac{2x+1}{2}}$$

But also

$$x = \frac{1}{2}\cos 2t$$
$$= \frac{1}{2}(1 - 2\sin^2 t)$$
$$\therefore \sin^2 t = \frac{1-2x}{2}$$
$$\sin t = \sqrt{\frac{1-2x}{2}}$$

Therefore

$$y = \sin\left(t + \frac{\pi}{6}\right)$$
$$y = \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$$
$$y = \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$$
$$y = \frac{\sqrt{3}}{2}\sqrt{\frac{1-2x}{2}} + \frac{1}{2}\sqrt{\frac{2x+1}{2}}$$
$$y = \sqrt{\frac{3-6x}{8}} + \sqrt{\frac{2x+1}{8}}$$
$$y^2 = \frac{1}{4}\left(\sqrt{3-12x^2} - 2x + 2\right)$$
$$4y^2 + 2x - 2 = \sqrt{3-12x^2}$$
$$\left(4y^2 + 2x - 2\right)^2 + 12x^2 - 3 = 0$$