Differentiation 9C

1 a $y=(1+2x)^4$

Let $u=1+2x$; then $y=u^4$

$$\frac{du}{dx}=2 \quad \text{and} \quad \frac{dy}{du}=4u^3$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=4u^3\times 2=8u^3 = 8(1+2x)^3$$

b $y=(3-2x^2)^{-5}$

Let $u=3-2x^2$; then $y=u^{-5}$

$$\frac{du}{dx}=-4x \quad \text{and} \quad \frac{dy}{du}=-5u^{-6}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(-5u^{-6})\times(-4x)=20xu^{-6}$$
$$=20x(3-2x^2)^{-6}$$

c $y=(3+4x)^{\frac{1}{2}}$

Let $u=3+4x$; then $y=u^{\frac{1}{2}}$

$$\frac{du}{dx}=4 \quad \text{and} \quad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times 4$$
$$=2u^{-\frac{1}{2}}=2(3+4x)^{-\frac{1}{2}}$$

d $y=(6x+x^2)^7$

Let $u=6x+x^2$; then $y=u^7$

$$\frac{du}{dx}=6+2x \quad \text{and} \quad \frac{dy}{du}=7u^6$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=7u^6\times(6+2x)$$
$$=7(6+2x)(6x+x^2)^6$$

e $y=\frac{1}{3+2x}=(3+2x)^{-1}$

Let $u=3+2x$; then $y=\frac{1}{u}=u^{-1}$

$$\frac{du}{dx}=2 \quad \text{and} \quad \frac{dy}{du}=-u^{-2}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-u^{-2}\times 2$$
$$=-2u^{-2}=\frac{-2}{(3+2x)^2}$$

f $y=\sqrt{7-x}=(7-x)^{\frac{1}{2}}$

Let $u=7-x$; then $y=u^{\frac{1}{2}}$

$$\frac{du}{dx}=-1 \quad \text{and} \quad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times(-1)$$
$$=-\frac{1}{2}(7-x)^{-\frac{1}{2}}=-\frac{1}{2\sqrt{7-x}}$$

1 g $y = 4(2+8x)^4$

Let $u = 2+8x$; then $y = 4u^4$

$\frac{du}{dx} = 8$ and $\frac{dy}{du} = 16u^3$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 16u^3 \times 8 = 128(2+8x)^3$

h $y = 3(8-x)^{-6}$

Let $u = 8-x$; then $y = 3u^{-6}$

$\frac{du}{dx} = -1$ and $\frac{dy}{du} = -18u^{-7}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -18u^{-7} \times (-1) = 18(8-x)^{-7}$

2 a $y = e^{\cos x}$

Let $u = \cos x$; then $y = e^u$

$\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = e^u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times (-\sin x) = -\sin x\, e^{\cos x}$

b $y = \cos(2x-1)$

Let $u = 2x-1$; then $y = \cos u$

$\frac{du}{dx} = 2$ and $\frac{dy}{du} = -\sin u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2 = -2\sin(2x-1)$

c $y = \sqrt{\ln x}$

Let $u = \ln x$; then $y = \sqrt{u} = u^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times \left(\frac{1}{x}\right)$

$= \frac{1}{2xu^{\frac{1}{2}}} = \frac{1}{2x\sqrt{\ln x}}$

d $y = (\sin x + \cos x)^5$

Let $u = \sin x + \cos x$; then $y = u^5$

$\frac{du}{dx} = \cos x - \sin x$ and $\frac{dy}{du} = 5u^4$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times (\cos x - \sin x)$

$= 5(\cos x - \sin x)(\sin x + \cos x)^4$

e $y = \sin(3x^2 - 2x + 1)$

Let $u = 3x^2 - 2x + 1$; then $y = \sin u$

$\frac{du}{dx} = 6x - 2$ and $\frac{dy}{du} = \cos u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times (6x-2)$

$= (6x-2)\cos(3x^2 - 2x + 1)$

f $y = \ln(\sin x)$

Let $u = \sin x$; then $y = \ln u$

$\frac{du}{dx} = \cos x$ and $\frac{dy}{du} = \frac{1}{u}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{\cos x}{\sin x} = \cot x$

g $y = 2e^{\cos 4x}$

Let $u = \cos 4x$; then $y = 2e^u$

$\frac{du}{dx} = -4\sin 4x$ and $\frac{dy}{du} = 2e^u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2e^u \times (-4\sin 4x)$

$= -8\sin 4x\, e^{\cos x}$

2 h $y = \cos(e^{2x} + 3)$

Let $u = e^{2x} + 3$; then $y = \cos u$

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dy}{du} = -\sin u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2e^{2x}$$
$$= -2e^{2x}\sin(e^{2x} + 3)$$

3 $y = \frac{1}{(4x+1)^2}$

Let $u = 4x + 1$; then $y = u^{-2}$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = -2u^{-3}$

$$\therefore \frac{dy}{dx} = -8u^{-3} = \frac{-8}{(4x+1)^3}$$

When $x = \frac{1}{4}$,

$$\frac{dy}{dx} = \frac{-8}{\left(4 \times \frac{1}{4} + 1\right)^3} = \frac{-8}{2^3} = -1$$

4 $y = (5 - 2x)^3$

Let $u = 5 - 2x$; then $y = u^3$

$\frac{du}{dx} = -2$ and $\frac{dy}{du} = 3u^2$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times (-2) = -6(5 - 2x)^2$$

When $x = 1$,

$y = 3^3 = 27$ and $\frac{dy}{dx} = -6 \times 3^2 = -54$

Equation of tangent at point P (1, 27) is

$y - 27 = -54(x - 1)$

or $y = -54x + 81$

5 $y = (1 + \ln 4x)^{\frac{3}{2}}$

Let $u = 1 + \ln 4x$; then $y = u^{\frac{3}{2}}$

$\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{3}{2}u^{\frac{1}{2}} \times \frac{1}{x} = \frac{3}{2x}\sqrt{1 + \ln 4x}$$

When $x = \frac{1}{4}e^3$,

$$\frac{dy}{dx} = \frac{3}{\frac{1}{2}e^3}\sqrt{1 + \ln e^3} = \frac{6}{e^3}\sqrt{1 + 3} = 12e^{-3}$$

6 a $x = y^2 + y$

$\frac{dx}{dy} = 2y + 1$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y + 1}$$

b $x = e^y + 4y$

$\frac{dx}{dy} = e^y + 4$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y + 4}$$

c $x = \sin 2y$

$\frac{dx}{dy} = 2\cos 2y$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2\cos 2y} = \frac{1}{2}\sec 2y$$

d $4x = \ln y + y^3$

$x = \frac{1}{4}\ln y + \frac{1}{4}y^3$

$\frac{dx}{dy} = \frac{1}{4y} + \frac{3}{4}y^2 = \frac{1 + 3y^3}{4y}$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{4y}{1 + 3y^3}$$

7 $x = 3y^2 - 2y$

$$\frac{dx}{dy} = 6y - 2$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{6y-2}$$

At (8, 2) the value of y is 2.

$$\therefore \frac{dy}{dx} = \frac{1}{12-2} = \frac{1}{10}$$

8 $x = y^{\frac{1}{2}} + y^{-\frac{1}{2}}$

$$\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}} - \frac{1}{2}y^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\frac{1}{2}y^{-\frac{1}{2}} - \frac{1}{2}y^{-\frac{3}{2}}}$$

At the point $\left(\frac{5}{2}, 4\right)$ the value of y is 4.

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{1}{2}(4)^{-\frac{1}{2}} - \frac{1}{2}(4)^{-\frac{3}{2}}} = \frac{1}{\frac{1}{2}\times\frac{1}{2} - \frac{1}{2}\times\frac{1}{8}}$$

$$= \frac{1}{\frac{1}{4} - \frac{1}{16}} = \frac{1}{\frac{3}{16}} = \frac{16}{3}$$

9 a $x = e^y$

$$\Rightarrow \frac{dx}{dy} = e^y$$

b $y = \ln x \Rightarrow e^y = x$

From part **a**, $\frac{dx}{dy} = e^y$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

10 a $x = 4\cos 2y$

When $x = 2$, $\cos 2y = \frac{1}{2}$

So $2y = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

$y = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

Hence $Q\left(2, \frac{\pi}{6}\right)$ lies on C.

b $\frac{dx}{dy} = -8\sin 2y \Rightarrow \frac{dy}{dx} = -\frac{1}{8\sin 2y}$

At Q, $y = \frac{\pi}{6}$

so $\frac{dy}{dx} = -\frac{1}{8\sin\frac{\pi}{3}} = -\frac{1}{8\times\frac{\sqrt{3}}{2}} = -\frac{1}{4\sqrt{3}}$

c Equation of normal to C at Q is

$$y - \frac{\pi}{6} = 4\sqrt{3}(x-2)$$

or $4\sqrt{3}x - y - 8\sqrt{3} + \frac{\pi}{6} = 0$

11 a $y = \sin^2 3x = (\sin 3x)^2$

Let $u = \sin 3x$; then $y = u^2$

$\frac{du}{dx} = 3\cos 3x$ and $\frac{dy}{du} = 2u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = 2u\times 3\cos 3x$$

$$= 6\sin 3x\cos 3x$$

b $y = e^{(x+1)^2}$

Let $u = (x+1)^2$; then $y = e^u$

$\frac{du}{dx} = 2(x+1)$ and $\frac{dy}{du} = e^u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = e^u\times 2(x+1) = 2(x+1)e^{(x+1)^2}$$

11 c $y = \ln(\cos x)^2$

Let $u = \cos x$; then $y = \ln u^2 = 2\ln u$

$\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = \frac{2}{u}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2}{u} \times (-\sin x)$$

$$= -2\frac{\sin x}{\cos x} = -2\tan x$$

d $y = \frac{1}{3+\cos 2x}$

Let $u = 3+\cos 2x$; then $y = \frac{1}{u} = u^{-1}$

$\frac{du}{dx} = -2\sin 2x$ and $\frac{dy}{du} = -u^{-2} = -\frac{1}{u^2}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times (-2\sin 2x)$$

$$= \frac{2\sin 2x}{(3+\cos 2x)^2}$$

e $y = \sin\left(\frac{1}{x}\right)$

Let $u = \frac{1}{x}$; then $y = \sin u$

$\frac{du}{dx} = -\frac{1}{x^2}$ and $\frac{dy}{du} = \cos u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times \left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

12 $y = \frac{4}{(2-4x)^2}$

Let $u = 2-4x$; then $y = \frac{4}{u^2} = 4u^{-2}$

$\frac{du}{dx} = -4$ and $\frac{dy}{du} = -8u^{-3} = -\frac{8}{u^3}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{8}{u^3} \times (-4) = \frac{32}{(2-4x)^3}$$

When $x = 3$, $y = \frac{4}{(-10)^2} = 0.04$

and $\frac{dy}{dx} = \frac{32}{(-10)^3} = -0.032$

Equation of normal at A is

$$y - 0.04 = \frac{1}{0.032}(x-3)$$

Multiplying through by 100 and rearranging gives

$$100y - 4 = 3125x - 9375$$

$$3125x - 100y - 9371 = 0$$

13 $y = 3^{x^3}$

Let $u = x^3$; then $y = 3^u$

$\frac{du}{dx} = 3x^2$ and $\frac{dy}{du} = 3^u \ln 3$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3^u \ln 3 \times 3x^2 = 3x^2 3^{x^3} \ln 3$$

When $x = 1$, $\frac{dy}{dx} = 3 \times 1^2 \times 3^{1^3} \times \ln 3 = 9\ln 3$

Challenge

a $y=\sqrt{\sin\sqrt{x}}$

Let $u=\sqrt{x}=x^{\frac{1}{2}}$; then $y=\sqrt{\sin u}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}$

Let $v=\sin u$; then $y=\sqrt{v}=v^{\frac{1}{2}}$

$\frac{\mathrm{d}v}{\mathrm{d}u}=\cos u$ and $\frac{\mathrm{d}y}{\mathrm{d}v}=\frac{1}{2}v^{-\frac{1}{2}}$

Using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{\mathrm{d}y}{\mathrm{d}v}\times\frac{\mathrm{d}v}{\mathrm{d}u}=\frac{1}{2}v^{-\frac{1}{2}}\times\cos u$$

$$=\frac{\cos u}{2y}=\frac{\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}$$

Using the chain rule again,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$$

b $\ln y=\sin^3(3x+4)$

Hence $y=\mathrm{e}^{\sin^3(3x+4)}$

Let $u=\sin(3x+4)$; then $y=\mathrm{e}^{u^3}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=3\cos(3x+4)$

Let $v=u^3$; then $y=\mathrm{e}^v$

$\frac{\mathrm{d}v}{\mathrm{d}u}=3u^2$ and $\frac{\mathrm{d}y}{\mathrm{d}v}=\mathrm{e}^v$

Using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{\mathrm{d}y}{\mathrm{d}v}\times\frac{\mathrm{d}v}{\mathrm{d}u}=\mathrm{e}^v\times 3u^2=3u^2\mathrm{e}^{u^3}$$

Using the chain rule again,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=3u^2\mathrm{e}^{u^3}\times 3\cos(3x+4)$$

$$=3\sin^2(3x+4)\mathrm{e}^{\sin^3(3x+4)}\times 3\cos(3x+4)$$

$$=9\mathrm{e}^{\sin^3(3x+4)}\cos(3x+4)\sin^2(3x+4)$$