Differentiation 9D

1 a Let $y = x(1+3x)^5$

Let $u = x$ and $v = (1+3x)^5$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 3\times 5(1+3x)^4$
(using the chain rule)

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x\times 15(1+3x)^4 + (1+3x)^5\times 1\\ &= (1+3x)^4(15x+1+3x)\\ &= (1+3x)^4(1+18x)\end{aligned}$$

b Let $y = 2x(1+3x^2)^3$

Let $u = 2x$ and $v = (1+3x^2)^3$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 18x(1+3x^2)^2$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\times 18x(1+3x^2)^2 + (1+3x^2)^3\times 2\\ &= (1+3x^2)^2\left(36x^2 + 2(1+3x^2)\right)\\ &= (1+3x^2)^2(42x^2+2)\\ &= 2(1+3x^2)^2(21x^2+1)\end{aligned}$$

c Let $y = x^3(2x+6)^4$

Let $u = x^3$ and $v = (2x+6)^4$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 8(2x+6)^3$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x^3\times 8(2x+6)^3 + (2x+6)^4\times 3x^2\\ &= x^2(2x+6)^3\left(8x+3(2x+6)\right)\\ &= x^2(2x+6)^3(14x+18)\\ &= x^2\times 2^3(x+3)^3\times 2(7x+9)\\ &= 16x^2(x+3)^3(7x+9)\end{aligned}$$

d Let $y = 3x^2(5x-1)^{-1}$

Let $u = 3x^2$ and $v = (5x-1)^{-1}$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 6x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -5(5x-1)^{-2}$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2\times\left(-5(5x-1)^{-2}\right) + (5x-1)^{-1}\times 6x\\ &= -15x^2(5x-1)^{-2} + 6x(5x-1)^{-1}\\ &= 3x(5x-1)^{-2}\left(-5x+2(5x-1)\right)\\ &= 3x(5x-2)(5x-1)^{-2}\end{aligned}$$

2 a Let $y = \mathrm{e}^{-2x}(2x-1)^5$

Let $u = \mathrm{e}^{-2x}$ and $v = (2x-1)^5$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2\mathrm{e}^{-2x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 10(2x-1)^4$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^{-2x}\times 10(2x-1)^4 + (2x-1)^5(-2\mathrm{e}^{-2x})\\ &= \mathrm{e}^{-2x}(2x-1)^4\left(10-2(2x-1)\right)\\ &= \mathrm{e}^{-2x}(2x-1)^4(12-4x)\\ &= -4(x-3)(2x-1)^4\mathrm{e}^{-2x}\end{aligned}$$

2 b Let $y = \sin 2x \cos 3x$

Let $u = \sin 2x$ and $v = \cos 3x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -3\sin 3x$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x(-3\sin 3x) + \cos 3x(2\cos 2x)$$
$$= 2\cos 2x\cos 3x - 3\sin 2x\sin 3x$$

c Let $y = \mathrm{e}^x \sin x$

Let $u = \mathrm{e}^x$ and $v = \sin x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x\cos x + \sin x\,\mathrm{e}^x = \mathrm{e}^x(\sin x + \cos x)$$

d Let $y = \sin(5x)\ln(\cos x)$

Let $u = \sin 5x$ and $v = \ln(\cos x)$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 5\cos 5x$

and $\frac{\mathrm{d}v}{\mathrm{d}x} = (-\sin x) \times \frac{1}{\cos x} = -\tan x$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(5x)(-\tan x) + 5\cos(5x)\ln(\cos x)$$
$$= 5\cos 5x\ln(\cos x) - \tan x\sin 5x$$

3 a $y = x^2(3x-1)^3$

Let $u = x^2$ and $v = (3x-1)^3$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 9(3x-1)^2$

Using the product rule $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \times 9(3x-1)^2 + (3x-1)^3 \times 2x$$
$$= x(3x-1)^2\big(9x + 2(3x-1)\big)$$
$$= x(3x-1)^2(15x-2) \qquad (*)$$

At the point (1, 8), $x = 1$.

Substituting $x = 1$ into expression (*):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times 2^2 \times 13 = 52$$

b $y = 3x(2x+1)^{\frac{1}{2}}$

Let $u = 3x$ and $v = (2x+1)^{\frac{1}{2}}$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$

and $\frac{\mathrm{d}v}{\mathrm{d}x} = 2 \times \frac{1}{2}(2x+1)^{-\frac{1}{2}} = (2x+1)^{-\frac{1}{2}}$

Using the product rule $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x(2x+1)^{-\frac{1}{2}} + 3(2x+1)^{\frac{1}{2}}$$
$$= 3(2x+1)^{-\frac{1}{2}}\big(x + (2x+1)\big)$$
$$= 3(3x+1)(2x+1)^{-\frac{1}{2}} \qquad (*)$$

At the point (4, 36), $x = 4$.

Substituting $x = 4$ into (*):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \times 13 \times 9^{-\frac{1}{2}} = 3 \times 13 \times \frac{1}{3} = 13$$

3 c $y=(x-1)(2x+1)^{-1}$

Let $u=x-1$ and $v=(2x+1)^{-1}$

Then $\frac{du}{dx}=1$ and $\frac{dv}{dx}=-2(2x+1)^{-2}$

Using the product rule $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=(x-1)\left(-2(2x+1)^{-2}\right)+(2x+1)^{-1}\times 1$$
$$=(2x+1)^{-2}\left(-2(x-1)+(2x+1)\right)$$
$$=3(2x+1)^{-2} \qquad (*)$$

At the point $\left(2,\frac{1}{5}\right)$, $x=2$.

Substituting $x=2$ into (*):

$$\frac{dy}{dx}=3\times 5^{-2}=\frac{3}{25}$$

4 $y=(x-2)^2(2x+3)$

Let $u=(x-2)^2$ and $v=(2x+3)$

Then $\frac{du}{dx}=2(x-2)$ and $\frac{dv}{dx}=2$

Using $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=(x-2)^2\times 2+(2x+3)\times 2(x-2)$$
$$=2(x-2)(x-2+2x+3)$$
$$=2(x-2)(3x+1)$$

At stationary points $\frac{dy}{dx}=0$

$2(x-2)(3x+1)=0$

$(x-2)=0$ or $(3x+1)=0$

$\therefore\ x=2$ or $-\frac{1}{3}$

$x=2\Rightarrow y=0$

$x=-\frac{1}{3}\Rightarrow y=\left(-\frac{7}{3}\right)^2\left(\frac{7}{3}\right)=\frac{343}{27}$

So the stationary points are

$(2,0)$ and $\left(-\frac{1}{3},\frac{343}{27}\right)$

5 $y=\left(x-\frac{\pi}{2}\right)^5\sin 2x$

Let $u=\left(x-\frac{\pi}{2}\right)^5$ and $v=\sin 2x$

$\frac{du}{dx}=5\left(x-\frac{\pi}{2}\right)^4$ and $\frac{dv}{dx}=2\cos 2x$

Using $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=\left(x-\frac{\pi}{2}\right)^5 2\cos 2x+\sin 2x\times 5\left(x-\frac{\pi}{2}\right)^4$$
$$=\left(x-\frac{\pi}{2}\right)^4\left(2\left(x-\frac{\pi}{2}\right)\cos 2x+5\sin 2x\right)$$

When $x=\frac{\pi}{4}$,

$$\frac{dy}{dx}=\left(-\frac{\pi}{4}\right)^4\left(2\left(-\frac{\pi}{4}\right)\cos\frac{\pi}{2}+5\sin\frac{\pi}{2}\right)$$
$$=\frac{\pi^4}{256}(0+5)=\frac{5\pi^4}{256}$$

6 $y = x^2\cos(x^2)$

Let $u = x^2$ and $v = \cos(x^2)$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -2x\sin(x^2)$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^2\left(-2x\sin(x^2)\right) + \cos(x^2)\times 2x$$
$$= 2x\left(\cos(x^2) - x^2\sin(x^2)\right)$$

When $x = \frac{\sqrt{\pi}}{2}$,

$$\frac{dy}{dx} = \sqrt{\pi}\left(\cos\frac{\pi}{4} - \frac{\pi}{4}\sin\frac{\pi}{4}\right)$$
$$= \sqrt{\pi}\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}}\left(1 - \frac{\pi}{4}\right)$$

Equation of tangent at $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ is

$$y - \frac{\pi\sqrt{2}}{8} = \sqrt{\frac{\pi}{2}}\left(1 - \frac{\pi}{4}\right)\left(x - \frac{\sqrt{\pi}}{2}\right)$$
$$8y - \pi\sqrt{2} = 4\sqrt{2\pi}\left(1 - \frac{\pi}{4}\right)\left(x - \frac{\sqrt{\pi}}{2}\right)$$
$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)\left(x - \frac{\sqrt{\pi}}{2}\right)$$
$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)x - \frac{\pi\sqrt{2}}{2}(4 - \pi)$$
$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2} + \frac{\pi\sqrt{2}}{2}(4 - \pi) = 0$$
$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2}\left(\frac{\pi - 2}{2}\right) = 0$$

This is in the form $ax + by + c = 0$ with

$a = \sqrt{2\pi}(\pi - 4)$, $b = 8$ and $c = -\pi\sqrt{2}\left(\frac{\pi - 2}{2}\right)$

7 $y = 3x^2(5x - 3)^3$

Let $u = 3x^2$ and $v = (5x - 3)^3$

$\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = 15(5x - 3)^2$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = 3x^2 \times 15(5x - 3)^2 + 6x(5x - 3)^3$$
$$= 3x(5x - 3)^2\left(15x + 2(5x - 3)\right)$$
$$= 3x(5x - 3)^2(25x - 6)$$

Hence $A = 3$, $n = 2$, $B = 25$ and $C = -6$.

8 a $y = (x + 3)^2 e^{3x}$

Let $u = (x + 3)^2$ and $v = e^{3x}$

$\frac{du}{dx} = 2(x + 3)$ and $\frac{dv}{dx} = 3e^{3x}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (x + 3)^2 \times 3e^{3x} + e^{3x} \times 2(x + 3)$$
$$= e^{3x}(x + 3)\left(3(x + 3) + 2\right)$$
$$= e^{3x}(x + 3)(3x + 11)$$

b When $x = 2$, $\frac{dy}{dx} = e^6 \times 5 \times 17 = 85e^6$

Hence the gradient at point C is $85e^6$.

9 a Let $y = (2\sin x - 3\cos x)\ln 3x$

Let $u = 2\sin x - 3\cos x$ and $v = \ln 3x$

Then $\frac{du}{dx} = 2\cos x + 3\sin x$ and $\frac{dv}{dx} = \frac{1}{x}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{2\sin x - 3\cos x}{x}$$
$$+ (2\cos x + 3\sin x)\ln 3x$$

9 b Let $y = x^4 e^{7x-3}$

Let $u = x^4$ and $v = e^{7x-3}$

Then $\dfrac{du}{dx} = 4x^3$ and $\dfrac{dv}{dx} = 7e^{7x-3}$

Using $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\frac{dy}{dx} = x^4 \times 7e^{7x-3} + 4x^3 e^{7x-3}$$

$$= x^3 e^{7x-3}(7x+4)$$

10 Let $y = x^5\sqrt{10x+6}$

Let $u = x^5$ and $v = \sqrt{10x+6} = (10x+6)^{\frac{1}{2}}$

Then $\dfrac{du}{dx} = 5x^4$

and $\dfrac{dv}{dx} = 10 \times \dfrac{1}{2}(10x+6)^{-\frac{1}{2}} = \dfrac{5}{\sqrt{10x+6}}$

Using $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\frac{dy}{dx} = \frac{5x^5}{\sqrt{10x+6}} + 5x^4\sqrt{10x+6}$$

When $x = 1$, $\dfrac{dy}{dx} = \dfrac{5}{\sqrt{16}} + 5\sqrt{16} = 21.25$

Challenge

a Let $y = e^x \sin^2 x \cos x$

Let $p = e^x$ and $q = \sin^2 x \cos x$

Then $y = pq$, $\dfrac{dp}{dx} = e^x$

and $\dfrac{dy}{dx} = p\dfrac{dq}{dx} + q\dfrac{dp}{dx}$

Let $u = \sin^2 x = (\sin x)^2$ and $v = \cos x$

Then $q = uv$,

$\dfrac{du}{dx} = 2\sin x \cos x$ and $\dfrac{dv}{dx} = -\sin x$

Using $\dfrac{dq}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\frac{dq}{dx} = \sin^2 x(-\sin x) + \cos x(2\sin x\cos x)$$

$$= -\sin^3 x + 2\sin x\cos^2 x$$

Using $\dfrac{dy}{dx} = p\dfrac{dq}{dx} + q\dfrac{dp}{dx}$

$$\frac{dy}{dx} = e^x(-\sin^3 x + 2\sin x\cos^2 x)$$

$$+ \sin^2 x\cos x \times e^x$$

$$= -e^x \sin x(\sin^2 x - 2\cos^2 x - \sin x\cos x)$$

Challenge

b Let $y = x(4x-3)^6(1-4x)^9$

Let $p = x(4x-3)^6$ and $q = (1-4x)^9$

Then $y = pq$, $\frac{dq}{dx} = -36(1-4x)^8$

and $\frac{dy}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx}$

Let $u = x$ and $v = (4x-3)^6$

Then $p = uv$,

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 24(4x-3)^5$

Using $\frac{dp}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dp}{dx} &= 24x(4x-3)^5 + (4x-3)^6\\ &= (4x-3)^5(24x+4x-3)\\ &= (4x-3)^5(28x-3)\end{aligned}$$

Using $\frac{dy}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= x(4x-3)^6\left(-36(1-4x)^8\right)\\ &\quad + (1-4x)^9(4x-3)^5(28x-3)\\ &= (4x-3)^5(1-4x)^8\\ &\quad \times\left(-36x(4x-3) + (1-4x)(28x-3)\right)\\ &= (4x-3)^5(1-4x)^8\\ &\quad \times\left(-144x^2 + 108x + 40x - 112x^2 - 3\right)\\ &= -(4x-3)^5(1-4x)^8(256x^2 - 148x + 3)\end{aligned}$$