Differentiation 9G

1 a $x = 2t,\ y = t^2 - 3t + 2$

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t - 3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-3}{2} = t - \frac{3}{2}$$

b $x = 3t^2,\ y = 2t^3$

$$\frac{dx}{dt} = 6t,\ \frac{dy}{dt} = 6t^2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2}{6t} = t$$

c $x = t + 3t^2,\ y = 4t$

$$\frac{dx}{dt} = 1 + 6t,\ \frac{dy}{dt} = 4$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4}{1+6t}$$

d $x = t^2 - 2,\ y = 3t^5$

$$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 15t^4$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{15t^4}{2t} = \frac{15t^3}{2}$$

e $x = \frac{2}{t},\ y = 3t^2 - 2$

$$\frac{dx}{dt} = -\frac{2}{t^2},\ \frac{dy}{dt} = 6t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{6t}{\frac{2}{t^2}} = -3t^3$$

f $x = \frac{1}{2t-1},\ y = \frac{t^2}{2t-1}$

$$\frac{dx}{dt} = -\frac{2}{(2t-1)^2},\ \frac{dy}{dt} = \frac{2t(2t-1) - 2t^2}{(2t-1)^2} = \frac{2t^2 - 2t}{(2t-1)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{2t^2 - 2t}{2} = t(1-t)$$

g $x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2}$

$$\frac{dx}{dt} = \frac{2(1+t^2) - 4t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4t}{2(1-t^2)} = \frac{2t}{t^2-1}$$

h $x = t^2 e^t,\ y = 2t$

$$\frac{dx}{dt} = t^2 e^t + 2te^t,\ \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{t^2 e^t + 2te^t} = \frac{2}{(t^2 + 2t)e^t}$$

i $x = 4\sin 3t,\ y = 3\cos 3t$

$$\frac{dx}{dt} = 12\cos 3t,\ \frac{dy}{dt} = -9\sin 3t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-9\sin 3t}{12\cos 3t} = -\frac{3}{4}\tan 3t$$

j $x = 2 + \sin t,\ y = 3 - 4\cos t$

$$\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = 4\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sin t}{\cos t} = 4\tan t$$

1 k $x=\sec t,\ y=\tan t$

$$\frac{dx}{dt}=\sec t\tan t,\ \frac{dy}{dt}=\sec^2 t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sec^2 t}{\sec t\tan t}=\frac{\sec t}{\tan t}=\text{cosec } t$$

l $x=2t-\sin 2t,\ y=1-\cos 2t$

$$\frac{dx}{dt}=2-2\cos 2t,\ \frac{dy}{dt}=2\sin 2t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2\sin 2t}{2-2\cos 2t}=\frac{\sin 2t}{1-\cos 2t}$$

$$=\frac{2\sin t\cos t}{1-(1-2\sin^2 t)}=\cot t$$

m $x=e^t-5,\ y=\ln t$

$$\frac{dx}{dt}=e^t,\ \frac{dy}{dt}=\frac{1}{t}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1}{te^t}$$

n $x=\ln t,\ y=t^2-64$

$$\frac{dx}{dt}=\frac{1}{t},\ \frac{dy}{dt}=2t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t}{\frac{1}{t}}=2t^2$$

o $x=e^{2t}+1,\ y=2e^t-1$

$$\frac{dx}{dt}=2e^{2t},\ \frac{dy}{dt}=2e^t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2e^t}{2e^{2t}}=\frac{1}{e^t}=e^{-t}$$

2 a $x=3-2\sin t,\ y=t\cos t$

$$\frac{dx}{dt}=-2\cos t,\ \frac{dy}{dt}=-t\sin t+\cos t$$

$$\therefore \frac{dy}{dx}=\frac{-t\sin t+\cos t}{-2\cos t}=\frac{t}{2}\tan t-\frac{1}{2}$$

At point P, where $t=\pi$,

$$x=3,\ y=-\pi \text{ and } \frac{dy}{dx}=-\frac{1}{2}$$

Equation of tangent is

$$y-(-\pi)=-\frac{1}{2}(x-3)$$

$$y=-\frac{1}{2}x+\frac{3}{2}-\pi$$

b $x=9-t^2,\ y=t^2+6t$

$$\frac{dx}{dt}=-2t,\ \frac{dy}{dt}=2t+6$$

$$\therefore \frac{dy}{dx}=-\frac{2t+6}{2t}=-\frac{t+3}{t}$$

At point P, where $t=2$,

$$x=5,\ y=16 \text{ and } \frac{dy}{dx}=-\frac{5}{2}$$

Equation of tangent is

$$y-16=-\frac{5}{2}(x-5)$$

$$2y-32=25-5x$$

$$2y+5x=57$$

3 a $x=e^t,\ y=e^t+e^{-t}$

$$\frac{dx}{dt}=e^t,\ \frac{dy}{dt}=e^t-e^{-t}$$

$$\therefore \frac{dy}{dx}=\frac{e^t-e^{-t}}{e^t}=1-e^{-2t}$$

At point P, where $t=0$,

$$\frac{dy}{dx}=1-1=0$$

Gradient of curve is 0

$\therefore$ normal is parallel to the y-axis.

When $t=0,\ x=1$ and $y=2$

Equation of the normal is $x=1$.

3 b $x = 1 - \cos 2t,\ y = \sin 2t$

$$\frac{dx}{dt} = 2\sin 2t,\ \frac{dy}{dt} = 2\cos 2t$$

$$\therefore \frac{dy}{dx} = \frac{2\cos 2t}{2\sin 2t} = \cot 2t$$

At point P, where $t = \frac{\pi}{6}$,

$$\frac{dy}{dx} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

$\therefore$ gradient of the normal is $-\sqrt{3}$

When $t = \frac{\pi}{6}$, $x = 1 - \cos\frac{\pi}{3} = \frac{1}{2}$

and $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Equation of the normal is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$$

$$y + \sqrt{3}x = \sqrt{3}$$

4 $x = \frac{t}{1-t},\ y = \frac{t^2}{1-t}$

Using the quotient rule,

$$\frac{dx}{dt} = \frac{(1-t)\times 1 - t(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dt} = \frac{(1-t)2t - t^2(-1)}{(1-t)^2} = \frac{2t - t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2t - t^2}{(1-t)^2} \div \frac{1}{(1-t)^2}$$
$$= 2t - t^2 = t(2-t)$$

$\frac{dy}{dx} = 0$ when $t = 0$ or 2

When $t = 0$, $x = 0$ and $y = 0$

When $t = 2$, $x = -2$ and $y = -4$

$\therefore$ (0, 0) and (−2, −4) are the points of zero gradient on the curve.

5 a $x = e^{2t},\ y = e^t - 1$

$$\frac{dx}{dt} = 2e^{2t},\ \frac{dy}{dt} = e^t$$

$$\therefore \frac{dy}{dx} = \frac{e^t}{2e^{2t}} = \frac{1}{2e^t}$$

When $t = \ln 2$,

$x = 4,\ y = 1$ and $\frac{dy}{dx} = \frac{1}{4}$

Equation of tangent is

$$y - 1 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x$$

b At stationary points $\frac{dy}{dx} = 0$

$$\frac{1}{2e^t} = 0 \Rightarrow e^{-t} = 0$$

This has no solutions, so the curve has no stationary points.

6 $x = \dfrac{t^2 - 3t - 4}{t},\ y = 2t$

$x = \dfrac{t^2}{t} - 3 - \dfrac{4}{t} = t - 3 - 4t^{-1}$

$\dfrac{dx}{dt} = 1 + \dfrac{4}{t^2},\ \dfrac{dy}{dt} = 2$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{1+\dfrac{4}{t^2}} = \dfrac{2t^2}{t^2+4}$

l_1 is parallel to $y = x + 5$

so gradient of l_1 is 1

$\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{2t^2}{t^2+4} = 1$

$\Rightarrow t^2 = 4$

so $t = 2$ (because $t > 0$)

When $t = 2$, $x = -3$ and $y = 4$

Equation of l_1 is

$y - 4 = 1 \times (x + 3)$

$y = x + 7$

7 a $x = 2\sin^2 t,\ y = 2\cot t$

$\dfrac{dx}{dt} = 4\sin t\cos t,\ \dfrac{dy}{dt} = -2\operatorname{cosec}^2 t$

$\therefore \dfrac{dy}{dx} = -\dfrac{2\operatorname{cosec}^2 t}{4\sin t\cos t} = -\dfrac{1}{2}\sec t\operatorname{cosec}^3 t$

b When $t = \dfrac{\pi}{6},\ x = \dfrac{1}{2}, y = 2\sqrt{3}$

and $\dfrac{dy}{dx} = -\dfrac{1}{2} \times \dfrac{2}{\sqrt{3}} \times 2^3 = -\dfrac{8}{\sqrt{3}} = -\dfrac{8\sqrt{3}}{3}$

Equation of tangent is

$y - 2\sqrt{3} = -\dfrac{8\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right)$

$\sqrt{3}y - 6 = -8x + 4$

$8x + \sqrt{3}y - 10 = 0$

8 a $x = 4\sin t,\ y = 2\operatorname{cosec} 2t$

$x = 2\sqrt{3} \Rightarrow 4\sin t = 2\sqrt{3}$

$\sin t = \dfrac{\sqrt{3}}{2} \quad \therefore t = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$t = \dfrac{\pi}{3} \Rightarrow y = 2\operatorname{cosec}\dfrac{2\pi}{3} = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$,

which is the y-coordinate of point A.

So $t = \dfrac{\pi}{3}$ at point A.

b $\dfrac{dx}{dt} = 4\cos t,\ \dfrac{dy}{dt} = -4\operatorname{cosec} 2t\cot 2t$

$\therefore \dfrac{dy}{dx} = -\dfrac{-4\operatorname{cosec} 2t\cot 2t}{4\cos t}$

$= -\dfrac{\cot 2t\operatorname{cosec} 2t}{\cos t}$

When $t = \dfrac{\pi}{3},\ \dfrac{dy}{dx} = -\dfrac{\cot\dfrac{2\pi}{3}\operatorname{cosec}\dfrac{2\pi}{3}}{\cos\dfrac{\pi}{3}}$

$= -\dfrac{\left(-\dfrac{1}{\sqrt{3}}\right) \times \dfrac{2}{\sqrt{3}}}{\dfrac{1}{2}} = \dfrac{4}{3}$

$\therefore$ gradient of normal is $-\dfrac{3}{4}$

Equation of normal, l, is

$y - \dfrac{4\sqrt{3}}{3} = -\dfrac{3}{4}(x - 2\sqrt{3})$

$12y - 16\sqrt{3} = -9(x - 2\sqrt{3})$

$9x + 12y - 34\sqrt{3} = 0$

9 a $x = t^2 + t,\ y = t^2 - 10t + 5$

$\dfrac{dx}{dt} = 2t + 1,\ \dfrac{dy}{dt} = 2t - 10$

$\therefore \dfrac{dy}{dx} = \dfrac{2t - 10}{2t + 1}$

When gradient is 2, $\dfrac{2t-10}{2t+1} = 2$

$2t - 10 = 4t + 2 \Rightarrow t = -6$

At P, $x = (-6)^2 - 6 = 30$

and $y = (-6)^2 - 10(-6) + 5 = 101$

Coordinates of P are (30, 101).

9 b Equation of tangent at P is

$y-101=2(x-30)$

$y=2x+41$

c Substituting for y and x in the tangent equation:

$t^2-10t+5=2(t^2+t)+41$

$t^2+12t+36=0$

Discriminant $=12^2-4\times 36=0$

Therefore the curve and the line only intersect once, so the tangent at P does not intersect the curve again.

10 a $x=2\sin t,\ y=\sqrt{2}\cos 2t$

$\frac{dx}{dt}=2\cos t,\ \frac{dy}{dt}=-2\sqrt{2}\sin 2t$

$\therefore \frac{dy}{dx}=\frac{-2\sqrt{2}\sin 2t}{2\cos t}=\frac{-\sqrt{2}\times 2\sin t\cos t}{\cos t}$

$=-2\sqrt{2}\sin t$

b When $t=\frac{\pi}{3}$:

$x=2\left(\frac{\sqrt{3}}{2}\right)=\sqrt{3},\ y=\sqrt{2}\left(-\frac{1}{2}\right)=-\frac{\sqrt{2}}{2}$

and $\frac{dy}{dx}=-2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right)=-\sqrt{6}$

Equation of normal at A is

$y-\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{\sqrt{6}}(x-\sqrt{3})$

$\sqrt{6}y+\sqrt{3}=x-\sqrt{3}$

$x-\sqrt{6}y-2\sqrt{3}=0$

c Substituting for y and x in the normal equation:

$2\sin t-\sqrt{6}\times\sqrt{2}\cos 2t-2\sqrt{3}=0$

$\sin t-\sqrt{3}\cos 2t-\sqrt{3}=0$

$\sin t-\sqrt{3}(1-2\sin^2 t)-\sqrt{3}=0$

$2\sqrt{3}\sin^2 t+\sin t-2\sqrt{3}=0$

$(2\sin t-\sqrt{3})(\sqrt{3}\sin t+2)=0$

$\sin t=\frac{\sqrt{3}}{2}$ or $\sin t=-\frac{2}{\sqrt{3}}$

(2nd option not possible since $|\sin t|\le 1$)

$\sin t=\frac{\sqrt{3}}{2}\Rightarrow t=\frac{\pi}{3}$ or $\frac{2\pi}{3}$

When $t=\frac{2\pi}{3}$:

$x=2\sin\frac{2\pi}{3}=\sqrt{3},\ y=\sqrt{2}\cos\frac{4\pi}{3}=-\frac{\sqrt{2}}{2},$

which is the same as point A, so l does not intersect C other than at point A.

11 a $x=\cos t,\ y=\frac{1}{2}\sin 2t$

$\frac{dx}{dt}=-\sin t,\ \frac{dy}{dt}=\cos 2t$

$\therefore \frac{dy}{dx}=-\frac{\cos 2t}{\sin t}$

b When $t=\frac{\pi}{6}$: $x=\frac{\sqrt{3}}{2},\ y=\frac{\sqrt{3}}{4}$

and $\frac{dy}{dx}=-\frac{\frac{1}{2}}{\frac{1}{2}}=-1$

Equation of tangent at A is

$y-\frac{\sqrt{3}}{4}=-\left(x-\frac{\sqrt{3}}{2}\right)$

i.e. $y=-x+\frac{3\sqrt{3}}{4}$

11 c l_1 and l_2 both have gradient -1

$\therefore$ values of t at points where the tangents cut the curve will be solutions to

$$-\frac{\cos 2t}{\sin t} = -1$$
$$1-2\sin^2 t = \sin t$$
$$2\sin^2 t + \sin t - 1 = 0$$
$$(2\sin t - 1)(\sin t + 1) = 0$$
$$\sin t = \frac{1}{2} \text{ or } -1$$
$$t = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

So lines l_1 and l_2 touch the curve when

$$t = \frac{5\pi}{6} \text{ and } t = \frac{3\pi}{2}.$$
$$t = \frac{5\pi}{6} \Rightarrow x = -\frac{\sqrt{3}}{2}, y = -\frac{\sqrt{3}}{4}$$

Equation of l_1 is

$$y - \left(-\frac{\sqrt{3}}{4}\right) = -1\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

i.e. $y = -x - \dfrac{3\sqrt{3}}{4}$

$$t = \frac{3\pi}{2} \Rightarrow x = 0, y = 0$$

Equation of l_2 is

$$y - 0 = -(x - 0)$$

i.e. $y = -x$