Differentiation 9H

1 $u = y^n$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = ny^{n-1}$$

$$\frac{\mathrm{d}(y^n)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$$

2 $$\frac{\mathrm{d}(xy)}{\mathrm{d}x} = x\frac{\mathrm{d}(y)}{\mathrm{d}x} + \frac{\mathrm{d}(x)}{\mathrm{d}x}y = x\frac{\mathrm{d}(y)}{\mathrm{d}x} + 1 \times y$$

$$= x\frac{\mathrm{d}y}{\mathrm{d}x} + y$$

3 a $x^2 + y^3 = 2$

Differentiate with respect to x:

$$2x + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{3y^2}$$

b $x^2 + 5y^2 = 14$

$$2x + 10y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{10y} = -\frac{x}{5y}$$

c $x^2 + 6x - 8y + 5y^2 = 13$

$$2x + 6 - 8\frac{\mathrm{d}y}{\mathrm{d}x} + 10y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$2x + 6 = (8 - 10y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+6}{8-10y} = \frac{x+3}{4-5y}$$

d $y^3 + 3x^2y - 4x = 0$

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + \left(3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y \times 6x\right) - 4 = 0$$

$$(3y^2 + 3x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - 6xy$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-6xy}{3(x^2+y^2)}$$

e $3y^2 - 2y + 2xy = x^3$

$$6y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + \left(2x\frac{\mathrm{d}y}{\mathrm{d}x} + y \times 2\right) = 3x^2$$

$$(6y - 2 + 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2y$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 2y}{2x + 6y - 2}$$

f $x = \dfrac{2y}{x^2 - y}$

$$x^3 - xy = 2y$$

$$x^3 - xy - 2y = 0$$

Differentiate with respect to x:

$$3x^2 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$3x^2 - y = (x+2)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - y}{x+2}$$

3 g $(x-y)^4 = x+y+5$

Differentiate with respect to x
(using the chain rule on the first term):

$$4(x-y)^3\left(1-\frac{dy}{dx}\right) = 1+\frac{dy}{dx}$$

$$4(x-y)^3 - 1 = \left(1+4(x-y)^3\right)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{4(x-y)^3-1}{1+4(x-y)^3}$$

h $e^x y = xe^y$

$$e^x\frac{dy}{dx} + ye^x = xe^y\frac{dy}{dx} + e^y \times 1$$

$$e^x\frac{dy}{dx} - xe^y\frac{dy}{dx} = e^y - ye^x$$

$$(e^x - xe^y)\frac{dy}{dx} = e^y - ye^x$$

$$\therefore \frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}$$

i $\sqrt{xy} + x + y^2 = 0$

$$(xy)^{\frac{1}{2}} + x + y^2 = 0$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\frac{dy}{dx} + y\right) + 1 + 2y\frac{dy}{dx} = 0$$

Multiply both sides by $2\sqrt{xy}$:

$$\left(x\frac{dy}{dx} + y\right) + 2\sqrt{xy} + 4y\sqrt{xy}\frac{dy}{dx} = 0$$

$$\left(x + 4y\sqrt{xy}\right)\frac{dy}{dx} = -2\sqrt{xy} - y$$

$$\therefore \frac{dy}{dx} = \frac{-2\sqrt{xy} - y}{x + 4y\sqrt{xy}}$$

4 $x^2 + 3xy^2 - y^3 = 9$

Differentiate with respect to x:

$$2x + \left(3x \times 2y\frac{dy}{dx} + y^2 \times 3\right) - 3y^2\frac{dy}{dx} = 0$$

Substitute $x = 2$ and $y = 1$ to give

$$4 + \left(12\frac{dy}{dx} + 3\right) - 3\frac{dy}{dx} = 0$$

$$9\frac{dy}{dx} = -7$$

$$\frac{dy}{dx} = -\frac{7}{9}$$

$\therefore$ gradient of the tangent at (2, 1) is $-\frac{7}{9}$

Equation of the tangent is

$$(y-1) = -\frac{7}{9}(x-2)$$

$$y = -\frac{7}{9}x + \frac{23}{9}$$

or $7x + 9y - 23 = 0$

5 $(x+y)^3 = x^2 + y$

Differentiate with respect to x:

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 2x + \frac{dy}{dx}$$

Substitute $x = 1$ and $y = 0$ to give

$$3\left(1+\frac{dy}{dx}\right) = 2 + \frac{dy}{dx}$$

$$2\frac{dy}{dx} = -1 \quad \therefore \frac{dy}{dx} = \frac{-1}{2}$$

$\therefore$ gradient of the normal at (1, 0) is 2.

Equation of the normal is

$$y - 0 = 2(x-1)$$

or $y = 2x - 2$

6 $x^2+4y^2-6x-16y+21=0$

Differentiate with respect to x:

$$2x+8y\frac{dy}{dx}-6-16\frac{dy}{dx}=0$$
$$8y\frac{dy}{dx}-16\frac{dy}{dx}=6-2x$$
$$(8y-16)\frac{dy}{dx}=6-2x$$
$$\frac{dy}{dx}=\frac{6-2x}{8y-16}$$

For zero gradient:

$$\frac{6-2x}{8y-16}=0$$
$$6-2x=0$$
$$x=3$$

Substitute $x=3$ into the equation of the curve to give

$$9+4y^2-18-16y+21=0$$
$$4y^2-16y+12=0$$
$$y^2-4y+3=0$$
$$(y-1)(y-3)=0$$
$$y=1 \text{ or } 3$$

$\therefore$ the coordinates of the points of zero gradient are (3, 1) and (3, 3).

7 $2x^2+3y^2-x+6xy+5=0$

$$4x+6y\frac{dy}{dx}-1+6\left(x\frac{dy}{dx}+y\right)=0$$
$$(6y+6x)\frac{dy}{dx}=1-6y-4x$$
$$\frac{dy}{dx}=\frac{1-6y-4x}{6(x+y)}$$

When $x=1$ and $y=-2$,

$$\frac{dy}{dx}=\frac{1-6(-2)-4}{6(1-2)}=-\frac{3}{2}$$

Equation of tangent at (1, −2) is

$$y-(-2)=-\frac{3}{2}(x-1)$$
$$2y+4=-3x+3$$
$$3x+2y+1=0$$

8 $3^x=y-2xy$

$$3^x\ln 3=\frac{dy}{dx}-2\left(x\frac{dy}{dx}+y\right)$$
$$3^x\ln 3+2y=(1-2x)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{3^x\ln 3+2y}{1-2x}$$

Substitute x = 2 and y = −3 to give

$$\frac{dy}{dx}=\frac{3^2\ln 3-6}{1-4}=2-3\ln 3$$

9 $\ln(y^2)=\frac{1}{2}x\ln(x-1)$

$$2\ln y=\frac{1}{2}x\ln(x-1)$$
$$\frac{2}{y}\frac{dy}{dx}=\frac{1}{2}\left(\left(x\times\frac{1}{x-1}\right)+\ln(x-1)\right)$$
$$\frac{dy}{dx}=\frac{y}{4}\left(\frac{x}{x-1}+\ln(x-1)\right)$$

When x = 4,
the equation of the curve gives

$$\ln(y^2)=2\ln 3=\ln 9\Rightarrow y^2=9$$
$$\therefore\ y=3\ \text{(because } y>0)$$

Hence $\dfrac{dy}{dx}=\dfrac{3}{4}\left(\dfrac{4}{3}+\ln 3\right)=1+\dfrac{3}{4}\ln 3$

10 a $\sin x + \cos y = 0.5$

$\cos x - \sin y \dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{\cos x}{\sin y}$

b At stationary points $\dfrac{dy}{dx} = 0$

$\dfrac{\cos x}{\sin y} = 0$ when $\cos x = 0$

$\therefore x = \pm\dfrac{\pi}{2}$ (in the interval $-\pi < x < \pi$)

When $x = \dfrac{\pi}{2}$, $1 + \cos y = 0.5$

$\cos y = -0.5 \Rightarrow y = \pm\dfrac{2\pi}{3}$

When $x = -\dfrac{\pi}{2}$, $-1 + \cos y = 0.5$

$\cos y = 1.5 \Rightarrow$ no solutions

Therefore the stationary points are $\left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$ and $\left(\dfrac{\pi}{2}, -\dfrac{2\pi}{3}\right)$.

11 a $ye^{-3x} - 3x = y^2$

$y(-3e^{-3x}) + e^{-3x}\dfrac{dy}{dx} - 3 = 2y\dfrac{dy}{dx}$

$(e^{-3x} - 2y)\dfrac{dy}{dx} = 3(ye^{-3x} + 1)$

$\dfrac{dy}{dx} = \dfrac{3(ye^{-3x} + 1)}{e^{-3x} - 2y}$

b Substitute $x = 0$ and $y = 0$ to give

$\dfrac{dy}{dx} = \dfrac{3(0 \times e^0 + 1)}{e^0 - 2 \times 0} = 3$

Equation of tangent at (0, 0) is

$y - 0 = 3(x - 0)$

$y = 3x$

Challenge

a $6x + y^2 + 2xy = x^2$

$6 + 2y\dfrac{dy}{dx} + 2\left(x\dfrac{dy}{dx} + y\right) = 2x$

$(2y + 2x)\dfrac{dy}{dx} = 2x - 2y - 6$

$\therefore \dfrac{dy}{dx} = \dfrac{2x - 2y - 6}{2y + 2x} = \dfrac{x - y - 3}{y + x}$

$\dfrac{dy}{dx} = 0$ only when $x - y - 3 = 0$

or $y = x - 3$

Substitute $y = x - 3$ into the equation of the curve to give

$6x + (x-3)^2 + 2x(x-3) = x^2$

$2x^2 - 6x + 9 = 0$

The discriminant of this quadratic is

$(-6)^2 - 4 \times 2 \times 9 = -36 < 0$

so there are no real solutions.

Hence there are no points on C such that $\dfrac{dy}{dx} = 0$.

b $\dfrac{dx}{dy} = \dfrac{y + x}{x - y - 3}$

$\dfrac{dx}{dy} = 0$ when $y + x = 0$

or $y = -x$

Substitute $y = -x$ into the equation of the curve to give

$6x + (-x)^2 + 2x(-x) = x^2$

$2x^2 - 6x = 0$

$x(x - 3) = 0$

$x = 0$ or $x = 3$

So $y = 0$ or $y = -3$

Therefore the coordinates of the points on C such that $\dfrac{dx}{dy} = 0$ are (0, 0) and (3, −3).