Differentiation 9J

1 $A = \frac{1}{4}\pi r^2$

$\frac{dA}{dr} = \frac{1}{2}\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= \frac{1}{2}\pi r \times 6 = 3\pi r$

When $r = 2$,

$\frac{dA}{dt} = 3\pi \times 2 = 6\pi$

2 $y = xe^x$

$\frac{dy}{dx} = xe^x + e^x = (x+1)e^x$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= (x+1)e^x \times 5 = 5(x+1)e^x$

When $x = 2$,

$\frac{dy}{dt} = 5(2+1)e^2 = 15e^2$

3 $r = 1 + 3\cos\theta$

$\frac{dr}{d\theta} = -3\sin\theta$

$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt}$

$= -3\sin\theta \times 3 = -9\sin\theta$

When $\theta = \frac{\pi}{6}$,

$\frac{dr}{dt} = -9\sin\frac{\pi}{6} = -\frac{9}{2}$

4 $V = \frac{1}{3}\pi r^3$

$\frac{dV}{dr} = \pi r^2 \Rightarrow \frac{dr}{dV} = \frac{1}{\pi r^2}$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\pi r^2} \times 8 = \frac{8}{\pi r^2}$

When $r = 3$,

$\frac{dr}{dt} = \frac{8}{\pi \times 3^2} = \frac{8}{9\pi}$

5 Let P be the size of the population and let t be time. Then the rate of growth of the population is $\frac{dP}{dt}$

$\frac{dP}{dt} \propto P$

i.e. $\frac{dP}{dt} = kP$ where k is a positive constant.

6 The gradient of the curve is $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} \propto xy$ (product of x- and y-coordinates)

i.e. $\frac{dy}{dx} = kxy$,

where k is the constant of proportion.

When $x = 4$, $y = 2$ and $\frac{dy}{dx} = \frac{1}{2}$.

Substituting into $\frac{dy}{dx} = kxy$ gives

$\frac{1}{2} = k \times 4 \times 2$

$k = \frac{1}{16}$

$\therefore \frac{dy}{dx} = \frac{xy}{16}$

7 The rate of increase of the volume of liquid in the container is $\frac{dV}{dt}$

$\frac{dV}{dt} = \text{rate in} - \text{rate out}$

$= 30 - \frac{2}{15}V$

Multiply both sides by –15 to give

$-15\frac{dV}{dt} = 2V - 450$

8 The rate of change of the charge is $\frac{dQ}{dt}$

$$\frac{dQ}{dt} \propto Q$$

i.e. $\frac{dQ}{dt} = -kQ$

where k is a positive constant.
(The negative sign is required as the body is *losing* charge.)

9 The rate of increase of x is $\frac{dx}{dt}$

$\therefore \frac{dx}{dt} \propto \frac{1}{x^2}$ (inverse proportion)

i.e. $\frac{dx}{dt} = \frac{k}{x^2}$

where k is the constant of proportion.

10 a Let r be the radius of the circle and let t be time. Then $\frac{dr}{dt} = 0.4 \text{ cm s}^{-1}$.

$$C = 2\pi r$$
$$\frac{dC}{dr} = 2\pi$$
$$\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$$
$$= 2\pi \times 0.4 = 0.8\pi \text{ cm s}^{-1}$$

This means that the circumference is increasing at a constant rate of 0.8π cm per second.

b Let A be the radius of the circle; then

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$= 2\pi r \times 0.4 = 0.8\pi r$$

When $r = 10$,

$$\frac{dA}{dt} = 0.8\pi \times 10 = 8\pi \text{ cm}^2 \text{ s}^{-1}$$

c $\frac{dA}{dt} = 0.8\pi r$

When $\frac{dA}{dt} = 20$,

$$0.8\pi r = 20$$
$$r = \frac{20}{0.8\pi} = \frac{25}{\pi} \text{ cm}$$

11 a Let l be the side length of the cube and let V be its volume.

Then $V = l^3$ and $\frac{dV}{dt} = -4.5$

$\frac{dV}{dl} = 3l^2$ so $\frac{dl}{dV} = \frac{1}{3l^2}$

$$\frac{dl}{dt} = \frac{dl}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{3l^2} \times (-4.5) = -\frac{3}{2l^2}$$

When $V = 100$, $l = \sqrt[3]{100}$

$$\frac{dl}{dt} = -\frac{3}{2\left(\sqrt[3]{100}\right)^2} = 0.070 \text{ cm s}^{-1} \text{ (2 s.f.)}$$

b $\frac{dl}{dt} = -\frac{3}{2l^2}$

2 mm is 0.2 cm

When $\frac{dl}{dt} = -0.2$,

$$-\frac{3}{2l^2} = -0.2$$
$$2l^2 = 15$$
$$l = \sqrt{7.5}$$
$$\therefore V = l^3 = \left(\sqrt{7.5}\right)^3 = 20.5 \text{ cm}^3 \text{ (3 s.f.)}$$

12 The rate of change of the volume of fluid in the tank is $\frac{dV}{dt}$

$\frac{dV}{dt} \propto \sqrt{V}$

i.e. $\frac{dV}{dt} = -K\sqrt{V}$

where K is a positive constant.
(The negative sign is present because fluid is flowing *out* of the tank, so the volume left in the tank is *decreasing*.)

Let A be the constant cross-section;
then $V = Ah$ (where h is the depth)

$\therefore \frac{dV}{dh} = A$

Use the chain rule to find $\frac{dh}{dt}$:

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$$

$$= \frac{-K\sqrt{V}}{A} = \frac{-K\sqrt{Ah}}{A}$$

$$= \left(\frac{-K}{\sqrt{A}}\right)\sqrt{h} = -k\sqrt{h}$$

where $k = \frac{K}{\sqrt{A}}$ is a positive constant.

13 a Let l be the length of one side of the cube.
Surface area of cube $A = 6l^2$.

So $l = \sqrt{\frac{A}{6}}$

Volume of cube $V = l^3 = \left(\sqrt{\frac{A}{6}}\right)^3 = \left(\frac{A}{6}\right)^{\frac{3}{2}}$

b $\frac{dV}{dA} = \frac{1}{6} \times \frac{3}{2}\left(\frac{A}{6}\right)^{\frac{1}{2}} = \frac{1}{4}\left(\frac{A}{6}\right)^{\frac{1}{2}}$

c Rate of expansion of surface area is $\frac{dA}{dt}$

Given $\frac{dA}{dt} = 2$

Need $\frac{dV}{dt}$ so use the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

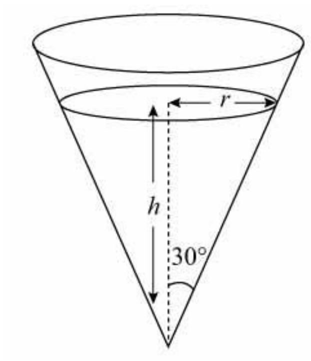
$$= \frac{1}{4}\left(\frac{A}{6}\right)^{\frac{1}{2}} \times 2 = \frac{1}{2}\left(\frac{A}{6}\right)^{\frac{1}{2}}$$

From $A = 6l^2$ and $V = l^3$,

$$A = 6\left(\sqrt[3]{V}\right)^2 = 6V^{\frac{2}{3}}$$

$$\therefore \frac{dV}{dt} = \frac{1}{2}\left(\frac{6V^{\frac{2}{3}}}{6}\right)^{\frac{1}{2}} = \frac{1}{2}V^{\frac{1}{3}}$$

14



Let V be the volume of salt in the funnel at time t.

$$V = \frac{1}{3}\pi r^2 h$$

$$\tan 30° = \frac{r}{h} \Rightarrow r = h\tan 30° = \frac{h}{\sqrt{3}}$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h = \frac{1}{9}\pi h^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi h^2 \text{ and hence } \frac{\mathrm{d}h}{\mathrm{d}V} = \frac{3}{\pi h^2}$$

Given that $\frac{\mathrm{d}V}{\mathrm{d}t} = -6$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3}{\pi h^2} \times (-6) = -\frac{18}{\pi h^2}$$

So the rate of change of the height, h, is inversely proportional to h^2 and is given by the differential equation $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{18}{\pi h^2}$