Review Exercise 1

1 Assumption: there are a finite number of prime numbers, p_1, p_2, p_3 up to p_n.

Let $X = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$

None of the prime numbers $p_1, p_2, p_3 \ldots p_n$ can be a factor of X as they all leave a remainder of 1 when X is divided by them. But X must have at least one prime factor. This is a contradiction. So there are infinitely many prime numbers.

2 Assumption: $x = \frac{a}{b}$ is a solution to the equation, $x^2 - 2 = 0$, where a and b are integers with no common factors.

$$\left(\frac{a}{b}\right)^2 - 2 = 0 \Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$$

So a^2 is even, which implies that a is even.
Write $a = 2n$ for some integer n.

$$(2n)^2 = 2b^2 \Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2 = b^2$$

So b^2 is even, which implies that b is even.
This contradicts the assumption that a and b have no common factor. Hence there are no rational solutions to the equation.

3 $$\frac{4x}{x^2-2x-3} + \frac{1}{x^2+x} = \frac{4x}{(x-3)(x+1)} + \frac{1}{x(x+1)}$$
$$= \frac{4x(x)+1(x-3)}{x(x+1)(x-3)} = \frac{4x^2+x-3}{x(x+1)(x-3)}$$
$$= \frac{(x+1)(4x-3)}{(x+1)x(x-3)} = \frac{4x-3}{x(x-3)}$$

4 **a** $$\mathrm{f}(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}$$
$$= \frac{(x+2)^2 - 3(x+2) + 3}{(x+2)^2}$$
$$= \frac{x^2+4x+4-3x-6+3}{(x+2)^2}$$
$$= \frac{x^2+x+1}{(x+2)^2}$$

4 b $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ ← Use the method of completing the square

$\geqslant\frac{3}{4}$ ← As $\left(x+\frac{1}{2}\right)^2\geqslant 0$

>0

for all values of x, $x\neq 2$

c $\mathrm{f}(x)=\frac{x^2+x+1}{(x+2)^2}$ from (a)

$\frac{x^2+x+1}{(x+2)^2}>0$

as $x^2+x+1>0$ from (b)

and $(x+2)^2>0$, for $x\neq -2$

So $\mathrm{f}(x)>0$, for $x\neq 2$

5 $\frac{2x-1}{(x-1)(2x-3)}=\frac{A}{x-1}+\frac{B}{2x-3}$

$\Rightarrow 2x-1=A(2x-3)+B(x-1)$

Set $x=1$: $2(1)-1=1=A(2(1)-3)=-A$

$\Rightarrow A=-1$

Set $\boldsymbol{x}=\frac{\mathbf{3}}{\mathbf{2}}$: $2\left(\frac{3}{2}\right)-1=2=B\left(\frac{3}{2}-1\right)=\frac{1}{2}B$

$\Rightarrow B=4$

So $\frac{2x-1}{(x-1)(2x-3)}=\frac{-1}{x-1}+\frac{4}{2x-3}$

6 $\frac{3x+7}{(x+1)(x+2)(x+3)}=\frac{P}{x+1}+\frac{Q}{x+2}+\frac{R}{x+3}$

$\Rightarrow 3x+7=P(x+2)(x+3)Q(x+1)(x+3)+R(x+1)(x+2)$

Set $x=-1$: $3(-1)+7=4=P((-1)+2)((-1)+3)=2P$

$\Rightarrow P=2$

Set $x=-2$: $3(-2)+7=1=Q((-2)+1)((-2)+3)=-Q$

$\Rightarrow Q=-1$

Set $x=-3$: $3(-3)+7=-2=R((-3)+1)((-3)+2)=2R$

$\Rightarrow R=-1$

So $P=2, Q=-1, R=-1$

7 $\dfrac{2}{(2-x)(1+x)^2} = \dfrac{A}{2-x} + \dfrac{B}{1+x} + \dfrac{C}{(1+x)^2}$

$\Rightarrow 2 = A(1+x)^2 + B(1+x)(2-x) + C(2-x)$

Set $x=2$: $2 = A(1+2)^2 = 9A$ so $A = \dfrac{2}{9}$

Set $x=-1$: $2 = C[2-(-1)] = 3C$ so $C = \dfrac{2}{3}$

Compare coefficients of x^2: $0 = A - B$

$\Rightarrow B = A = \dfrac{2}{9}$

Solution: $A = \dfrac{2}{9}, B = \dfrac{2}{9}, C = \dfrac{2}{3}$

8 $\dfrac{14x^2+13x+2}{(x+1)(2x+1)^2} \equiv \dfrac{A}{x+1} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2}$

You need denominators of $(x+1)$, $(2x+1)$ and $(2x+1)^2$

$\equiv \dfrac{A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)}{(x+1)(2x+1)^2}$

Add the three fractions

Compare numerators of fractions

$14x^2+13x+2 = A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)$

Set the numerators equal

Put $x=-1$

$3 = A + 0 + 0 \Rightarrow A = 3$

To find A set $x=-1$

Put $x = -\dfrac{1}{2}$

$\dfrac{14}{4} - \dfrac{13}{2} + 2 = \dfrac{1}{2}C \Rightarrow C = -2$

To find C set $x = -\frac{1}{2}$

So $14x^2+13x+2 = 3(2x+1)^2 + B(x+1)(2x+1) - 2(x+1)$

Compare coefficients of x^2:

$14 = 12 + 2B \Rightarrow B = 1$

Equate terms in x^2

$14x^2 = 3(2x)^2 + 2Bx^2$

Check constant term

$2 = 3 + 1 - 2$

Solve equation to find B

So $\dfrac{14x^2+13x+2}{(x+1)(2x+1)^2} \equiv \dfrac{3}{x+1} + \dfrac{1}{2x+1} - \dfrac{2}{(2x+1)^2}$

9 $\dfrac{3x^2+6x-2}{x^2+4} \equiv d + \dfrac{ex+f}{x^2+4}$

$\Rightarrow 3x^2+6x-2 = d(x^2+4)+ex+f$

Compare coefficients of x^2: $3 = d$

Compare coefficients of x: $6 = e$

Compare constant terms: $-2 = 4d + f$

So $f = -2-4d = -2-4(3) = -14$

Solution: $d = 3$, $e = 6$, $f = -14$

10 $\text{p}(x) = \dfrac{9-3x-12x^2}{(1-x)(1+2x)} = A + \dfrac{B}{1-x} + \dfrac{C}{1+2x}$

$\Rightarrow 9-3x-12x^2 = A(1-x)(1+2x)+B(1+2x)+C(1-x)$

Set $x = 1$: $9-3(1)-12(1)^2 = -6 = B(1+2(1)) = 3B$

$\Rightarrow B = -2$

Set $x = -\dfrac{1}{2}$: $9-3\left(-\dfrac{1}{2}\right)-12\left(-\dfrac{1}{2}\right)^2 = \dfrac{15}{2} = C\left(1-\left(-\dfrac{1}{2}\right)\right) = \dfrac{3}{2}C$

$\Rightarrow C = 5$

Compare coefficients of x^2: $-12 = -2A$

$\Rightarrow A = 6$

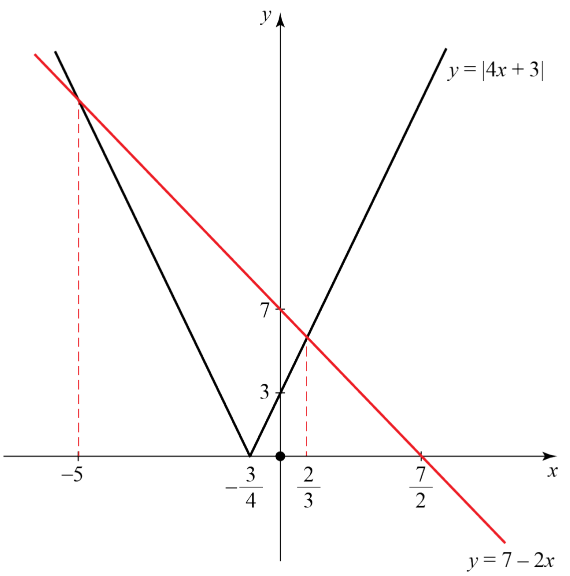
Solution: $A = 6$, $B = -2$, $C = 5$

11 First solve $|4x-3| = 7-2x$

$x > -\dfrac{3}{4}$: $4x+3 = 7-2x \Rightarrow x = \dfrac{2}{3}$

$x < -\dfrac{3}{4}$: $-(4x+3) = 7-2x \Rightarrow x = -5$

Now draw the lines $y = |4x+3|$ and $y = 7-2x$



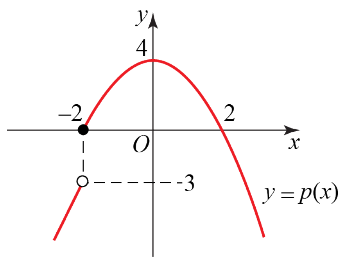
From the graph, we see that $|4x+3| > 7-2x$ when $x < -5$ or $x > \dfrac{2}{3}$

12 a For $x < -2$, p(x) is a straight line with gradient 4.

At $x = -2$, there is a discontinuity. $\text{p}(-2) = 0$ so draw an open dot at $(-2, -3)$ where the line section ends and a solid dot at $(-2,\ 0)$ where p(x) is defined.

For $x > -2$, $\text{p}(x) = 4 - x^2$. There is a maximum at (0, 4) since $x^2 \geqslant 0$, and the curve intersects the x-axis at (2, 0) since $4 - x^2 = 0 \Rightarrow x = \pm 2$

From the diagram, the range is $\text{p}(x) \leqslant 4$



b $\text{p}(a) = -20$

Check both sections of the domain for solutions.

$x < -2$: $4x + 5 = -20 \Rightarrow x = -\dfrac{25}{4}$

This is less than –2 so it is a solution.

$x \geqslant -2$: $4 - x^2 = -20 \Rightarrow x = \pm 2\sqrt{6}$

But $-2\sqrt{6} < -2$ so discard this possibility; $a = 2\sqrt{6} \geqslant 2$ so is a solution

Solutions are $a = -\dfrac{25}{4}, a = 2\sqrt{6}$

13 a
$$\text{qp}(x) = 2\left(\frac{1}{x+4}\right) - 5$$
$$= \frac{2}{x+4} - \frac{5(x+4)}{x+4}$$
$$= \frac{2 - 5x - 20}{x+4}$$
$$= \frac{-5x - 18}{x+4}$$

So $\text{qp}(x) = \dfrac{-5x - 18}{x+4}, x \in \mathbb{R}, x \neq -4$

Solutions are: $a = -5, b = -18, c = 1, d = 4$

b $\text{qp}(x) = 15$
$$\Rightarrow \frac{-5x - 18}{x+4} = 15$$
$$-5x - 18 = 15(x+4) = 15x + 60$$
$$-5x - 18 = 15x + 60$$
$$20x = -78$$
$$x = -\frac{39}{10}$$

13 c Let $y = \mathrm{r}(x)$

$$y = \frac{-5x-18}{x+4}$$

$$y(x+4) = -5x-18$$

$$x(y+5) = -4y-18$$

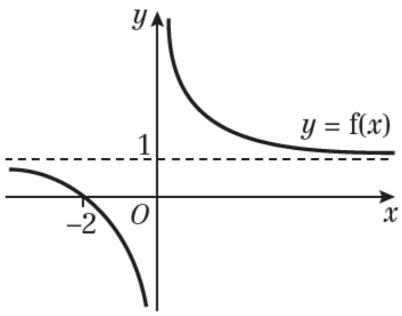
$$x = \frac{-4y-18}{y+5}$$

So $\mathrm{r}^{-1}(x) = \frac{-4x-18}{x+5}, \quad x \in \mathbb{R}, x \neq -5$

14 a $\frac{x+2}{x} = 1 + \frac{2}{x}$

Sketch $y = \frac{1}{x}$, stretch by a factor of 2

in the y-direction, translate by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



14 b $\mathrm{f}^2(x) = \mathrm{f}\left(\frac{x+2}{x}\right)$

$$= \frac{\frac{x+2}{2} + 2}{\frac{x+2}{x}}$$

$$\frac{\frac{x+2+2x}{x}}{\frac{x+2}{x}}$$

$$= \frac{(3x+2)}{x} \times \frac{x}{(x+2)}$$

$$= \frac{3x+2}{x+2}$$

So $\mathrm{f}^2(x) = \frac{3x+2}{x+2}, x \in \mathbb{R}, x \neq 0, x \neq -2$

c $\mathrm{gf}\left(\frac{1}{4}\right) = \mathrm{g}\left(\frac{2\frac{1}{4}}{\frac{1}{4}}\right) = \mathrm{g}(9)$

$$= \ln(18-5)$$

$$= \ln 13$$

14 d Let $y = \ln(2x-5)$

$$e^y = 2x - 5$$

$$\Rightarrow x = \frac{e^y + 5}{2}$$

$$g^{-1}(x) = \frac{e^x + 5}{2}, \quad x \in \mathbb{R}$$

The range of g(x) is $x \in \mathbb{R}$ so the domain of $g^{-1}(x)$ is $x \in \mathbb{R}$

15 a $pq(x) = 3(1-2x) + b = 3 + b - 6x$

$qp(x) = 1 - 2(3x + b) = 1 - 2b - 6x$

As $pq(x) = qp(x)$

$\Rightarrow 3 + b - 6x = 1 - 2b - 6x$

$\Rightarrow b = -\frac{2}{3}$

b Let $y = p(x)$

$y = 3x - \frac{2}{3}$

$\Rightarrow x = \frac{2 + 3y}{9}$

$p^{-1}(x) = \frac{3x + 2}{9}, x \in \mathbb{R}$

Let $z = q(x)$

$z = 1 - 2x$

$\Rightarrow x = \frac{1 - z}{2}$

$q^{-1}(x) = \frac{1 - x}{2}, x \in \mathbb{R}$

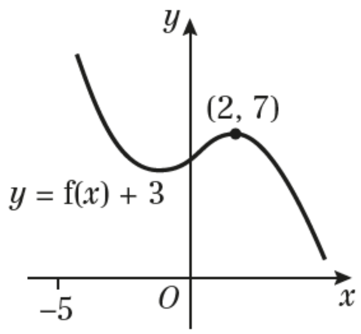
c $p^{-1}q^{-1}(x) = \dfrac{2 + 3\left(\frac{1-x}{2}\right)}{9} = \dfrac{-3x + 7}{18}, x \in \mathbb{R}$

$q^{-1}p^{-1}(x) = \dfrac{1 - \frac{2 + 3x}{9}}{2} = \dfrac{-3x + 7}{18}, x \in \mathbb{R}$

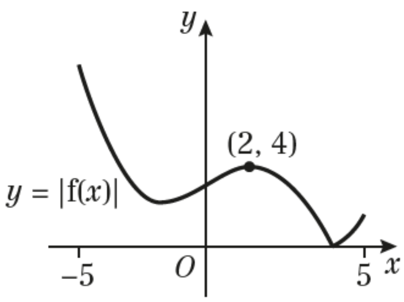
So $p^{-1}q^{-1}(x) = q^{-1}p^{-1}(x)$

And $a = -3, b = 7, c = 18$

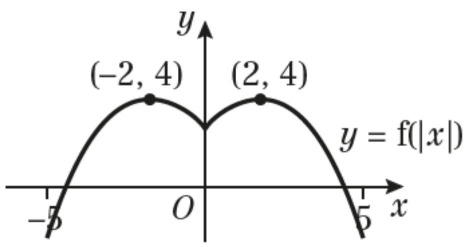
16 a Translation of $+3$ in the y direction. The maximum turning point is (2, 7).



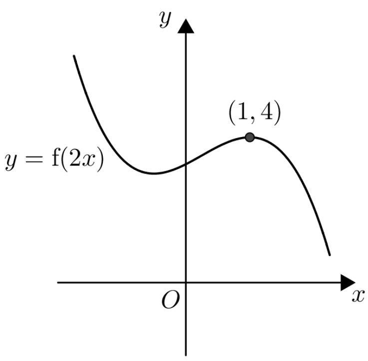
b For $y \geq 0$, curve is $y = \text{f}(x)$
For $y < 0$, reflect in x-axis.
The maximum turning point is (2, 4)



c For $x < 0$, $\text{f}|x| = \text{f}(-x)$, so draw $y = \text{f}(x)$ for $x \geqslant 0$, and then reflect this in $x = 0$
The maximum turning points are (–2, 4) and (2, 4)



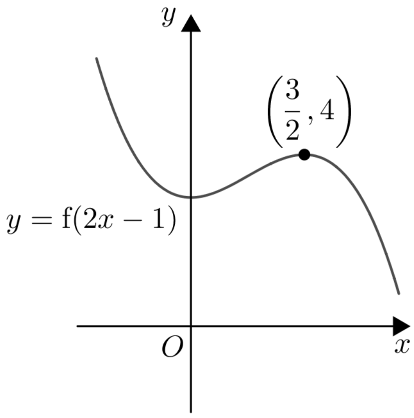
d $y = \text{f}(2x-1)$ can be written as $y = \text{f}(2(x-\frac{1}{2}))$
$y = \text{f}(2x)$
Horizontal stretch, scale factor $\frac{1}{2}$.



16 d (continued)

$y = f(2(x - \frac{1}{2}))$

Horizontal translation of $+\frac{1}{2}$



17 a To find intersections with the x-axis, solve h(x) =0

$2(x+3)^2 - 8 = 0$

$\Rightarrow (x+3)^2 = 4$

$\Rightarrow x = -3 \pm 2$

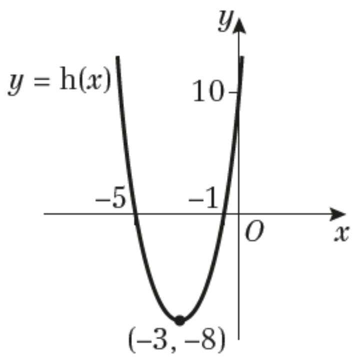
So there are intersections at (–5, 0) and (–1, 0)

To find intersections with the y-axis, find h(0)

$h(0) = 2(3)^2 - 8 = 10$

So there is an intersection at (0, 10)

Since (x + 3)2 ⩾ 0, there is a turning point (minimum) at (–3, –8)



17 b i $y = 3\text{h}(x+2)$

$\Rightarrow y = 3\left(2\left(x+2+3\right)^2 - 8\right)$

$\Rightarrow y = 6(x+5)^2 - 24$

This has a turning point when $x = -5$ at $(-5, -24)$

ii $y = \text{h}(-x)$

$\Rightarrow y = 2(-x+3)^2 - 8$

$\Rightarrow y = 2(3-x)^2 - 8$

This has a turning point when $x = 3$ at $(3, -8)$

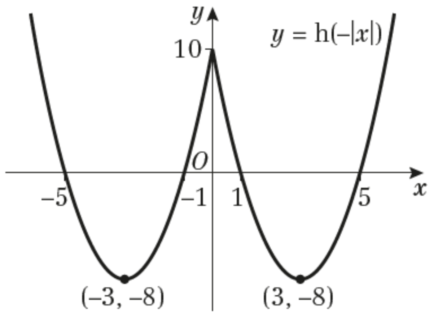
iii The modulus of $\text{h}(x)$ is the curve in part (a), with the section for $-5 < x < -1$ reflected in the x-axis. The turning point is $(-3, 8)$

c On one graph, reflect $\text{h}(x)$ in the y-axis to see what $\text{h}(-x)$ looks like.

Now to obtain the sketch of $\text{h}(-|x|)$, start a new graph,

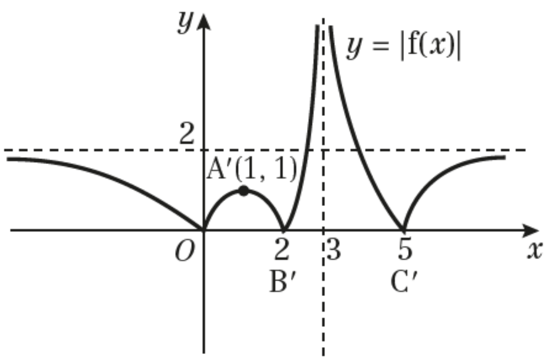
copy $\text{h}(-x)$ for $x \geqslant 0$, then reflect the result in the y-axis.

The x-intercepts are $(-5, 0)$, $(-1, 0)$, $(1, 0)$, $(5, 0)$; the y-intercept is $(0, 10)$ and there are minimum turning points at $(-3,-8)$ and $(3,-8)$.



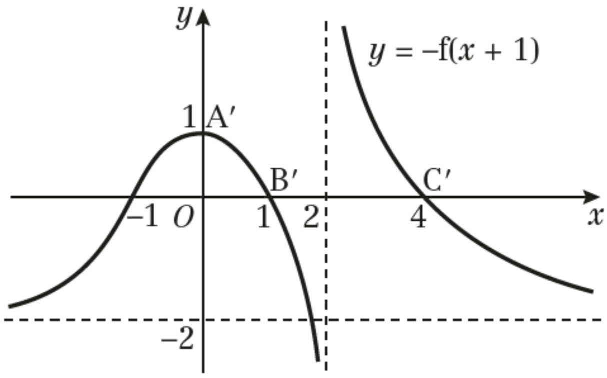
18 a i All parts of curve $y = \text{f}(x)$ below the x-axis are reflected in x-axis.

$A \to (1,1)$, B and C do not move.



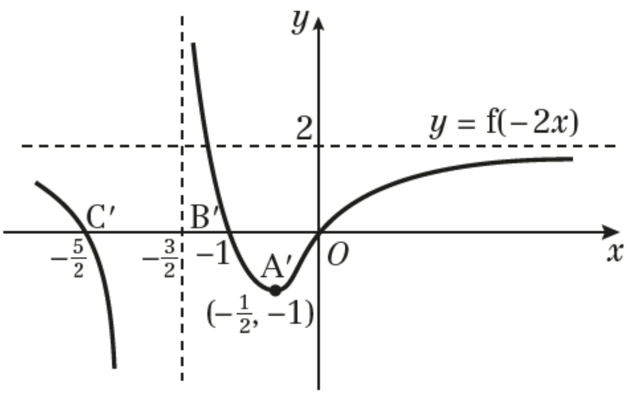
18 a ii Translate by -1 in the x direction and reflect in the x-axis.

$A \to (0, 1), B \to (1, 0), C \to (4, 0)$



iii Stretch in the x direction with scale factor $\frac{1}{2}$ and reflect in the y-axis.

$A \to (-\frac{1}{2}, -1), B \to (-1, 0), C \to (-\frac{5}{2}, 0)$



b i $3|f(x)| = 2 \Rightarrow |f(x)| = \frac{2}{3}$

Number of solutions is 6

ii $2|f(x)| = 3 \Rightarrow |f(x)| = \frac{3}{2}$

Number of solutions is 4

Consider graph **a i**

i How many times does the line $y = \frac{2}{3}$ cross the curve?

Line is below A'

ii Draw the line $y = \frac{3}{2}$

19 a $q(x) = \frac{1}{2}|x+b| - 3$

$q(0) = \frac{|b|}{2} - 3 = \frac{3}{2} \Rightarrow |b| = 9$

$b < 0$ so $b = -9$

b A is $(9, -3)$

To find B:

$x > 9$ so solve $\frac{1}{2}(x-9) - 3 = 0$

$\Rightarrow x = 15$

So B is (15, 0)

c $q(x) = \frac{1}{2}|x-9| - 3 = -\frac{x}{3} + 5$

$x < 9$: $\frac{9-x}{2} - 3 = -\frac{x}{3} + 5$

$3(9-x) - 18 = -2x + 30$

$27 - 18 - 30 = x$

$x = -21$

$x > 9$: $\frac{x-9}{2} - 3 = -\frac{x}{3} + 5$

$3(x-9) - 18 = -2x + 30$

$5x = 27 + 18 + 30$

$5x = 75$

$x = 15$

Solution set; –21, 15

20 a $-\frac{5}{3}|x+4| \leqslant 0 \Rightarrow$ range is $f(x) \leqslant 8$

b Over the whole domain, f(x) is not a one-one function so it cannot have an inverse.

20 c First solve $-\frac{5}{3}|x+4|+8=\frac{2}{3}x+4$

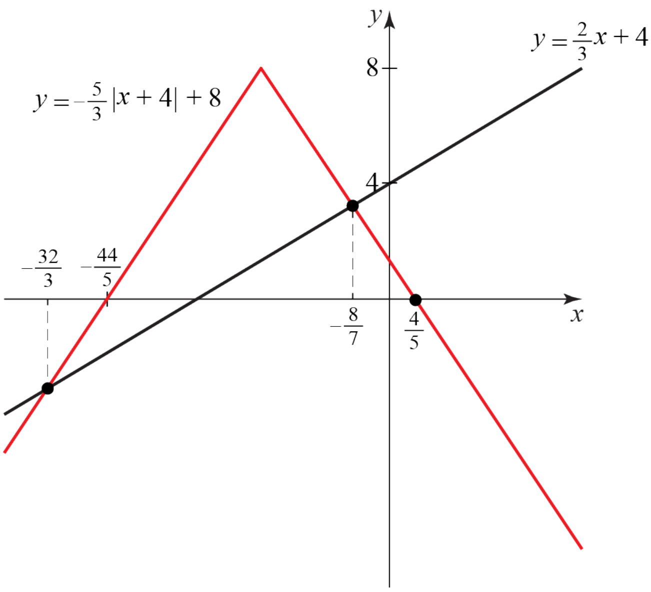
$x<4:\ \frac{5}{3}(x+4)+8=\frac{2}{3}x+4$

$$5(x+4)+24=2x+12$$
$$3x=12-24-20$$
$$x=-\frac{32}{3}$$

$x>4:\ -\frac{5}{3}(x+4)+8=\frac{2}{3}x+4$

$$-5(x+4)+24=2x+12$$
$$7x=-20+24-12$$
$$x=-\frac{8}{7}$$

Now sketch the lines $y=-\frac{5}{3}|x+4|+8$ and $y=\frac{2}{3}x+4$



From the graph we see that the inequality is satisfied in the region

$$-\frac{32}{3}<x<-\frac{8}{7}$$

d From the sketch drawn from part (c), the equation will have no solutions if the line lies above the apex of f(x) at (–4, 8)

$$\Rightarrow \frac{5}{3}(-4)+k>8$$
$$\Rightarrow k>8+\frac{20}{3}$$
$$\Rightarrow k>\frac{44}{3}$$

21 a $12-7k+d=3k^2 \Rightarrow 3k^2+7k-12=d$

$3k^2+d=k^2-10k \Rightarrow -2k^2-10k=d$

Subtracting the second equation from the first gives

$5k^2+17k-12=(5k-3)(k+4)=0$

So $k=\frac{3}{5}=0.6$ or $k=-4$

b Since the sequence contains only integer terms, $k=-4$.

$u_4=12-7(-4)=40$, $u_5=3(-4)^2=48$

So common difference d is $d=u_5-u_4=48-40=8$

The first term a satisfies

$a+3d=u_4 \Rightarrow a=40-3(8)=16$

So $a=16$, $d=8$

22 a First find the common difference and first term.

$u_4=a+3d=72 \quad (1)$

$u_{11}=a+10d=51 \quad (2)$

$(1)-(2): \ -7d=21 \Rightarrow d=-3$

Into (1): $a=72-3(-3)=81$

Now, using $S_n=\frac{n}{2}(2a+(n-1)d)$

$S_n=\frac{n}{2}(2(81)+(n-1)(-3))=1125$

$\Rightarrow n(162-3n+3)=2250$

$\Rightarrow -3n^2+165n=2250$

$\Rightarrow 3n^2-165n+2250=0$

b $3n^2-165n+2250=0$

$\Rightarrow n^2-55n+750=0$

$\Rightarrow (n-25)(n-30)=0$

$\Rightarrow n=25, n=30$

23 a $a = 19p - 18$

$d = u_2 - a = (17p - 8) - (19p - 18) = 10 - 2p$

So $u_{30} = a + 29d = (19p - 18) + 29(-2p + 10)$

$u_{30} = 272 - 39p$

b $S_{31} = \frac{31}{2}(2a + (31 - 1)d) = 0$

$\Rightarrow 2a + 30d = 0$

So $2(19p - 18) + 30(10 - 2p) = 0$

$(38 - 60)p - 36 + 300 = 0$

$22p = 264$

$p = 12$

24 a $u_2 = ar = 256, u_8 = ar^7 = 900$

$\frac{ar^7}{ar} = \frac{900}{256}$

$\Rightarrow r^6 = \frac{225}{64}$

$\Rightarrow \ln r^6 = \ln\left(\frac{225}{64}\right)$

$\Rightarrow 6\ln r - \ln\left(\frac{225}{64}\right) = 0$ (as $\ln x^k = k\ln x$)

$\Rightarrow 6\ln r + \ln\left(\frac{64}{225}\right) = 0$ (as $\ln x^{-1} = -\ln x$)

b Noting $r > 1$, so r is positive

$r = \left(\frac{225}{64}\right)^{\frac{1}{6}} = 1.2331060... = 1.23$ (3 s.f.)

25 a $r = \frac{ar}{a} = \frac{u_2}{u_1} = \frac{ar}{r}$

So $r = \frac{\frac{50}{6}}{10} = \frac{5}{6}$

$\therefore$ As $|r| < 1$, $S_\infty = \frac{a}{1 - r} = \frac{10}{1 - \frac{5}{6}} = 60$

25 b $a = 10, r = \frac{5}{6}$

$$S_k = \frac{10\left(1-\left(\frac{5}{6}\right)^k\right)}{1-\frac{5}{6}}$$

As $S_k > 55 \Rightarrow 1-\left(\frac{5}{6}\right)^k > \frac{55}{60}$

$$\Rightarrow 1-\left(\frac{5}{6}\right)^k > \frac{11}{12}$$

$$\Rightarrow \frac{1}{12} > \left(\frac{5}{6}\right)^k \Rightarrow \log\left(\frac{1}{12}\right) > \log\left(\frac{5}{6}\right)^k$$

$$\Rightarrow \log\left(\frac{1}{12}\right) > k\log\left(\frac{5}{6}\right)$$

$$\Rightarrow k > \frac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)}$$

(the inequality reverses direction in the final step because $\ln\frac{5}{6} < 0$)

c k must be a positive integer.

$\frac{\ln\frac{1}{12}}{\ln\frac{5}{6}} = 13.629$ (3 d.p.)

So the minimum value of k is 14.

26 a $4,\ 4r,\ 4r^2, \ldots$

$4 + 4r + 4r^2 = 7$

$4r^2 + 4r - 3 = 0$ (as required)

Use ar^{n-1} to write down expressions for the first three terms. Here a = 4 and n = 1, 2, 3

b $4r^2 + 4r - 3 = 0$

$(2r-1)(2r+3) = 0$

$r = \frac{1}{2},\ r = -\frac{3}{2}$

Factorise $4r^2 + 4r - 3 = -12$

$(-2) + (+6) = +4$, so

$4r^2 - 2r + 6r - 3 = 2r(2r-1) + 3(2r-1)$

$= (2r-1)(2r+3)$

c $r = \frac{1}{2}$

$$\frac{a}{1-r} = \frac{4}{1-\frac{1}{2}}$$

$$= \frac{4}{\frac{1}{2}}$$

$$= 8$$

Use $S_\infty = \frac{a}{1-r}$

Here $a = 4$ and $r = \frac{1}{2}$

27 a $ar^3 = x,\ ar^4 = 3,\ ar^5 = x+8$

$$\frac{ar^5}{ar^4} = \frac{ar^4}{ar^3}$$

so $\frac{x+8}{3} = \frac{3}{x}$

$x(x+8) = 9$

$x^2 + 8x - 9 = 0$

$(x+9)(x-1) = 0$

$x = 1,\ x = -9$

$r = \frac{ar^4}{ar^3} = \frac{3}{x}$

When $x = 1,\ r = 3$

When $x = -9,\ r = -\frac{1}{3}$

> $\frac{ar^5}{ar^4} = r$ and $\frac{ar^4}{ar^3} = r$ so $\frac{ar^5}{ar^4} = \frac{ar^4}{ar^3}$
>
> Clear the fractions. Multiply each side by $3x$
>
> so that $3x \times \frac{x+8}{3} = x(x+8)$ and $3x \times \frac{3}{x} = 9$

> Find r. Substitute $x = 1$, then $x = -9$, into $r = \frac{ar^4}{ar^3} = \frac{3}{x}$

b $r = -\frac{1}{3}$

$ar^4 = 3$

$a\left(-\frac{1}{3}\right)^4 = 3$

$a = 243$

> Remember $S_\infty = \frac{a}{1-r}$ for $|r| < 1$, so $r = -\frac{1}{3}$

c $S_\infty = \frac{a}{1-r} = \frac{243}{1+\frac{1}{3}} = 182.25$

28 a $a_{n+1} = 3a_n + 5$

$n = 1: a_2 = 3a_1 + 5$

$a_2 = 3k + 5$

> Use the given formula with $n = 1$

b $n = 2:\ a_3 = 3a_2 + 5$

$= 3(3k+5) + 5$

$= 9k + 15 + 5$

$= 9k + 20$

c i $\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$

$n = 3: a_4 = 3a_3 + 5$

$= 3(9k+20) + 5$

$= 27k + 65$

$\sum_{r=1}^{4} a_r = k + (3k+5) + (9k+20) + (27k+65)$

$= 40k + 90$

> This is *not* an arithmetic series.

> You cannot use a standard formula, so work out each separate term and then add them together to find the required sum.

28 c ii $\sum_{r=1}^{4} a_r = 10(4k+9)$

There is a factor of 10, so the sum is divisible by 10.

Give a conclusion.

29 a $a = 2400,\ r = 1.06$

After 4 years,

$2400(1.06)^3 = 2858.44\ldots = 2860$ to the nearest 10.

b $2400 \times 1.06^{N-1} > 6000 \Rightarrow 1.06^{N-1} > 2.5$

$\Rightarrow \log 1.06^{N-1} > \log 2.5 \Rightarrow (N-1)\log 1.06 > \log 2.5$

c Rearranging the inequality

$$N > \frac{\ln 2.5}{\ln 1.06} + 1 = 16.7 \text{ (1 d.p.)}$$

So $N = 17$

d The total amount raised is $5(S_{10})$

$$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{2400(1-1.06^{10})}{1.106} = 31633.90 \text{ (2 d.p.)}$$

Therefore the total amount raised is 5 × 31633.9, which to the nearest £1000 is £158,000

30 a Common ratio is $r = -4x$

Condition for the convergence of infinite sum is

$|r| < 1 \Rightarrow |-4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$

b $\sum_{r=1}^{\infty} 6 \times (-4x)^{r-1} = S_\infty = \frac{24}{5}$

Another equation for S_∞ is $S_\infty = \frac{a}{1-r} = \frac{6}{1+4x}$

So $\frac{6}{1+4x} = \frac{24}{5}$

$\Rightarrow 30 = 24 + 96x$

$\Rightarrow x = \frac{6}{96} = \frac{1}{16}$

31 a Using the binomial expansion

$$g(x) = (1-x)^{-\frac{1}{2}}$$

$$= 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)}{3!}(-x)^3 + \ldots$$

$$= 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \ldots$$

b $|x| < 1$

32 a $(1+ax)^n \equiv 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \ldots$

$na = -6 \qquad (1)$

$\frac{n(n-1)}{2}a^2 = 45 \qquad (2)$

Set coefficient of x, from binomial expansion, equal to –6 and set coefficient of x^2 equal to 45

From equation (1) $\quad a = -\frac{6}{n}$

Substitute into equation (2)

$$\frac{n(n-1)}{2} \times \frac{36}{n^2} = 45$$

Eliminate a from the simultaneous equations to obtain an equation in one variable n

$$36n^2 - 36n = 90n^2$$

$$-36n = 54n^2$$

Solve to find non-zero value for n

$\Rightarrow n = 0$ or $n = -\frac{36}{54} = -\frac{2}{3}$

Substitute into equation (1) to give $a = 9$

Check solutions in equation (2)

b Coefficient of $x^3 = \frac{n(n-1)(n-2)}{3!}a^3$

$$= \frac{-\frac{2}{3} \times -\frac{5}{3} \times -\frac{8}{3} \times 9^3}{3!}$$

$$= \frac{-80 \times 27}{6}$$

$$= -360$$

Substitute values found for n and a into the binomial expansion to give the coefficient of x^3

c The expansion is valid if $|9x| < 1$

So $-\frac{1}{9} < x < \frac{1}{9}$

The terms in the expansion are $(9x)$, $(9x)^2$, $(9x)^3 \ldots$ and so $|9x| < 1$

33 a Using the binomial expansion

$$(1+4x)^{\frac{3}{2}} = 1+\left(\frac{3}{2}\right)(4x)+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(4x)^2+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(4x)^3+\ldots$$

$$= 1+6x+6x^2-4x^3+\ldots$$

b $$\left(1+4\left(\frac{3}{100}\right)\right)^{\frac{3}{2}} = \left(\frac{112}{100}\right)^{\frac{3}{2}}$$

$$= \left(\sqrt{\frac{112}{100}}\right)^3$$

$$= \frac{112\sqrt{122}}{1000}$$

c $$1+6\left(\frac{3}{100}\right)+6\left(\frac{3}{100}\right)^2-4\left(\frac{3}{100}\right)^3 = 1.185292$$

So $\dfrac{112\sqrt{112}}{1000} \approx 1.185292$

$$\Rightarrow \sqrt{112} \approx \frac{1185.292}{112} = 10.582962857\ldots = 10.58296 \text{ (5 d.p.)}$$

d Using a calculator $\sqrt{112} = 10.5830052$ (7 d.p.)

$$\text{Percentage error } = \frac{10.5830052-10.5829643}{10.5830052}\times 100 = 0.00039\% \text{ (5 d.p.)}$$

Note, you will get different answers if you use values rounded to 5 d.p. in calculating the percentage error.

34 Expand $(3+2x)^{-3}$ using the binomial expansion:

$$(3+2x)^{-3} = 3^{-3}\left(1+\frac{2}{3}x\right)^{-3}$$

$$= \frac{1}{27}\left(1+(-3)\left(\frac{2}{3}x\right)+\frac{(-3)(-4)}{2!}\left(\frac{2}{3}x\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{2}{3}x\right)^3+\ldots\right)$$

$$= \frac{1}{27}\left(1-2x+\frac{8}{3}x^2-\frac{80}{27}x^3+\ldots\right)$$

So $(1+x)(3+2x)^{-3} = \frac{1}{27}(1+x)\left(1-2x+\frac{8}{3}x^2-\frac{80}{27}x^3+\ldots\right)$

$$= \frac{1}{27}\left(1+(-2+1)x+\left(\frac{8}{3}-2\right)x^2+\left(-\frac{80}{27}+\frac{8}{3}\right)x^3+\ldots\right)$$

$$= \frac{1}{27}-\frac{1}{27}x+\frac{2}{81}x^2-\frac{8}{729}x^3+\ldots$$

35 a $h(x) = (4-9x)^{\frac{1}{2}} = 2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$

So using the binomial expansion

$$h(x) = 2\left(1+\left(\frac{1}{2}\right)\left(-\frac{9}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{9}{4}x\right)^2+\ldots\right)$$

$$= 2\left(1-\frac{9}{8}x-\frac{81}{128}x^2+\ldots\right)$$

$$= 2-\frac{9}{4}x-\frac{81}{64}x^2+\ldots$$

b $h\left(\frac{1}{100}\right) = \left(4-\frac{9}{100}\right)^{\frac{1}{2}} = \left(\frac{400-9}{100}\right)^{\frac{1}{2}} = \frac{\sqrt{391}}{10}$

c $h\left(\frac{1}{100}\right) \approx 2-\frac{9}{4}\left(\frac{1}{100}\right)-\frac{81}{64}\left(\frac{1}{100}\right)^2 = 1.97737$ (5 d.p.)

36 a $(a+bx)^{-2} = \frac{1}{a^2}\left(1+\frac{b}{a}x\right)^{-2}$

$$= \frac{1}{a^2}\left(1+(-2)\left(\frac{b}{a}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{b}{a}x\right)^2+\ldots\right)$$

$$= \frac{1}{a^2}-\frac{2b}{a^3}x+\frac{3b^2}{a^4}x^2+\ldots$$

$$= \frac{1}{4}+\frac{1}{4}x+cx^2\ldots$$

So $\frac{1}{a^2}=\frac{1}{4} \Rightarrow a^2=4 \Rightarrow a=\pm 2$

When $a=2$, comparing the x coefficient gives

$-\frac{2b}{a^3}=\frac{1}{4} \Rightarrow b=-\frac{a^3}{8}=-1$

Comparing the x^2 coefficient gives

$c=\frac{3b^2}{a^4}=\frac{3}{2^4}=\frac{3}{16}$

So one solution is $a=2,\ b=-1,\ c=\frac{3}{16}$

When $a=-2$, comparing the x coefficient gives

$-\frac{2b}{a^3}=\frac{1}{4} \Rightarrow b=-\frac{a^3}{8}=1$

Comparing the x^2 coefficient gives

$c=\frac{3b^2}{a^4}=\frac{3}{2^4}=\frac{3}{16}$

So second solution is $a=-2,\ b=1,\ c=\frac{3}{16}$

Note that the two solutions yield the same expression

$(2-x)^{-2}=\left(-1\times(x-2)\right)^{-2}=(-1)^{-2}(x-2)^{-2}=(x-2)^{-2}$

b Coefficient of x^3 in expansion of $(x-2)^{-2}$

$\frac{1}{4}\frac{(-2)(-3)(-4)}{3!}\left(-\frac{1}{2}\right)^3=\frac{1}{8}$

37 a $\frac{3+5x}{(1+3x)(1-x)}=\frac{A}{1+3x}+\frac{B}{1-x}$

$\Rightarrow 3+5x=A(1-x)+B(1+3x)$

Set $x=1$: $8=4B \Rightarrow B=2$

Set $x=-\frac{1}{3}$: $\frac{4}{3}=\frac{4}{3}A \Rightarrow A=1$

37 b $\dfrac{3+5x}{(1+3x)(1-x)} = (1+3x)^{-1} + 2(1-x)^{-1}$

$= \left(1+(-1)(3x)+\dfrac{(-1)(-2)}{2!}(3x)^2+\ldots\right)+2\left(1+(-1)(-x)+\dfrac{(-1)(-2)}{2!}(-x)^2+\ldots\right)$

$= (1+2)+(-3x+2x)+(9x^2+2x^2)+\ldots$

$= 3-x+11x^2+\ldots$

38 a $\dfrac{3x-1}{(1-2x)^2} = \dfrac{A}{1-2x}+\dfrac{B}{(1-2x)^2}$

$\Rightarrow 3x-1 = A(1-2x)+B$

Set $x=\dfrac{1}{2}$: gives B= $\dfrac{1}{2}$

Compare coefficients of x gives $3=-2A \Rightarrow A=-\dfrac{3}{2}$

b $\dfrac{3x-1}{(1-2x)^2} = -\dfrac{3}{2}(1-2x)^{-1}+\dfrac{1}{2}(1-2x)^{-2}$

Expand each term using the bionomial expansion

$-\dfrac{3}{2}(1-2x)^{-1} = -\dfrac{3}{2}\left(1+(-1)(-2x)+\dfrac{(-1)(-2)}{2!}(-2x)^2+\dfrac{(-1)(-2)(-3)}{3!}(-2x)^3+\ldots\right)$

$\dfrac{1}{2}(1-2x)^{-2} = \dfrac{1}{2}\left(1+(-2)(-2x)+\dfrac{(-2)(-3)}{2!}(-2x)^2+\dfrac{(-2)(-3)(-4)}{3!}(-2x)^3+\ldots\right)$

Now sum the expansions

$-\dfrac{3}{2}(1-2x)^{-1}+\dfrac{1}{2}(1-2x)^{-2} = \left(-\dfrac{3}{2}+\dfrac{1}{2}\right)+\left(-3x+2x\right)+(-6x^2+6x^2)+(-12x^3+16x^3)+\ldots$

$= -1-x+4x^3+\ldots$

39 a $\mathrm{f}(x) = \dfrac{25}{(3+2x)^2(1-x)} = \dfrac{A}{3+2x}+\dfrac{B}{(3+2x)^2}+\dfrac{C}{1-x}$

$\Rightarrow 25 = A(3+2x)(1-x)+B(1-x)+C(3+2x)^2$

Set $x=1$: $25=25C \Rightarrow C=1$

Set $x=-\dfrac{3}{2}$: $25=\dfrac{5}{2}B \Rightarrow B=10$

Compare the coefficients of x^2

$0=-2A+4C \Rightarrow A=2C=2$

So $A=2, B=10, C=1$

39 b From part (a) $f(x) = 2(3+2x)^{-1} + 10(3+2x)^{-2} + (1-x)^{-1}$

$$= \frac{2}{3}\left(1+\frac{2}{3}x\right)^{-1} + \frac{10}{9}\left(1+\frac{2}{3}x\right)^{-2} + (1-x)^{-1}$$

Now expand each part of the equation using the binomial expansion

$$f(x) = \frac{2}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^2+\ldots\right)+\frac{10}{9}\left(1+(-2)\left(\frac{2}{3}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{2}{3}x\right)^2+\ldots\right)$$

$$+\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\ldots\right)$$

$$= \left(\frac{2}{3}+\frac{10}{9}+1\right)+\left(-\frac{4}{9}x-\frac{40}{27}x+x\right)+\left(\frac{8}{27}x^2+\frac{40}{27}x^2+x^2\right)+\ldots$$

$$= \frac{25}{9}-\frac{25}{27}x+\frac{25}{9}x^2+\ldots$$

40 a $\dfrac{40x^2+30x+31}{(x+4)(2x+3)} = A + \dfrac{B}{x+4} + \dfrac{C}{2x+3}$

$\Rightarrow 4x^2+30x+31 = A(x+4)(2x+3)+B(2x+3)+C(x+4)$

Set $x=-4$: $64-120+31=-25=-5B \Rightarrow B=5$

Set $x=-\dfrac{3}{2}$: $9-45+31=-5=\dfrac{5}{2}C \Rightarrow C=-2$

Compare coefficients of x^2

$4=2A \Rightarrow A=2$

Solution: $A=2,\ B=5,\ C=-2$

b $2+5(x+4)^{-1}-2(2x+3)^{-1}$

Rewrite as $f(x) = 2+\dfrac{5}{4}\left(1+\dfrac{x}{4}\right)^{-1}-\dfrac{2}{3}\left(1+\dfrac{2}{3}x\right)^{-1}$

$$f(x) = 2+\frac{5}{4}\left(1+(-1)\left(\frac{x}{4}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{4}\right)^2+\ldots\right)-\frac{2}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^2+\ldots\right)$$

$$= \left(2+\frac{5}{4}-\frac{2}{3}\right)+\left(-\frac{5}{16}x+\frac{4}{9}x\right)+\left(\frac{5}{64}x^2-\frac{8}{27}x^2\right)+\ldots$$

$$= \frac{31}{12}+\frac{19}{144}x-\frac{377}{1728}x^2+\ldots$$

Challenge

1 a B is located where $\mathbf{g}(\boldsymbol{x}) = -\frac{3}{4}\boldsymbol{x} + \frac{3}{2} = 0 \Rightarrow \boldsymbol{x} = 2$

So B has coordinates $(2, 0)$

To find A solve $f(x) = g(x)$ for $x < -3$

$3(x+3)+15 = -\frac{3}{4}x + \frac{3}{2}$

$\Rightarrow 12x + 96 = -3x + 6$

$\Rightarrow 15x = -90$

$\Rightarrow x = -6$

$g(-6) = f(-6) = 6$

So A has coordinates $(-6, 6)$

M is the midpoint of A and so has coordinates $\left(\frac{-6+2}{2}, \frac{6+0}{2}\right) = (-2, 3)$

To find the radius of the circle, use Pythagoras' theorem to find the length of MA:

$|MA| = \sqrt{(2-(-2))^2 + (3-0)^2} = \sqrt{25} = 5$

Therefore the equation of the circle is

$(x+2)^2 + (y-3)^2 = 25$

b For $x < -3,\ f(x) = 3(x+3)+15 = 3x+24$

Substituting $y = 3x + 24$ into the equation of the circle

$(x+2)^2 + (3x+21)^2 = (x+2)^2 + 9(x+7)^2 = 25$

$\Rightarrow 10x^2 + 130x + 420 = 0$

$\Rightarrow x^2 + 13x + 42 = 0$

$\Rightarrow (x+7)(x+6) = 0$

Solutions $x = -7, x = -7$

From the diagram, at P $x = -7$, and $f(x) = -12 + 15 = 3$

So P has coordinates $(-7, 3)$

Angle $\angle APB = 90^o$ by circle theorems so the area of the triangle is $\frac{1}{2}|AP||PB|$

$|AP| = \sqrt{1^2 + 3^2} = \sqrt{10}$

$|PB| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}$

$\text{Area} = \frac{1}{2}(\sqrt{10})(3\sqrt{10}) = 15$

2 The general term of the sequence is

$$a_m = m + (m-1)k$$

$$\Rightarrow \sum_{i=6}^{11} a_i = 6m + (5+6+7+8+9+10)k = 6m + 45k$$

$$\Rightarrow \sum_{i=12}^{15} a_i = 4m + (11+12+13+14)k = 4m + 50k$$

So $6m + 45k = 4m + 50k \Rightarrow m = \frac{5}{2}k$

3 $\mathrm{p}(x) = |x^2 - 8x + 12| = |(x-6)(x-2)|$

$\mathrm{q}(x) = |x^2 - 11x + 28| = |(x-4)(x-7)|$

To find the x-coordinate of A solve

$$-x^2 + 8x - 12 = x^2 - 11x + 28$$

$$\Rightarrow 2x^2 - 19x + 40 = 0$$

$$\Rightarrow x = \frac{19 - \sqrt{361 - 4(2)(40)}}{2(2)} = \frac{19 - \sqrt{41}}{4}$$

Using the quadratic formula, and from the graph we know to take the negative square root.

To find the x-coordinate of B solve

$$-x^2 + 8x - 12 = -x^2 + 11x - 28$$

$$\Rightarrow x = \frac{16}{3}$$

To find the x-coordinate of C solve

$$x^2 - 8x + 12 = -x^2 + 11x - 28$$

$$\Rightarrow 2x^2 - 19x + 40 = 0$$

$$\Rightarrow x = \frac{19 + \sqrt{41}}{4}$$

Taking the positive square root this time.

Solution is $A: \frac{19 - \sqrt{41}}{4}, B: \frac{16}{3}, C: \frac{19 + \sqrt{41}}{4}$

4 $$\sum_{r=1}^{40} \log_3\left(\frac{2n+1}{2n-1}\right) = \log_3\left(\frac{3}{1}\right) + \log_3\left(\frac{5}{3}\right) + \ldots + \log_3\left(\frac{79}{77}\right) + \log_3\left(\frac{81}{79}\right)$$

$$= \log_3\left(\frac{3}{1} \times \frac{5}{3} \times \ldots \times \frac{79}{77} \times \frac{81}{79}\right)$$

$$= \log_3 81$$

$$= 4$$

5 $y = f(ax+b)$ is a stretch by horizontal scale factor $\frac{1}{a}$ followed by a translation $\begin{pmatrix} -\frac{b}{a} \\ 0 \end{pmatrix}$.

Point (x, y) maps to point $\left(\frac{x}{a} - \frac{b}{a}, y\right)$.

So (x, y) invariant implies that: $\frac{x}{a} - \frac{b}{a} = x \Rightarrow x = \frac{b}{1-a}$