Review Exercise 2

1 Crosses y-axis when $x = 0$ at $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

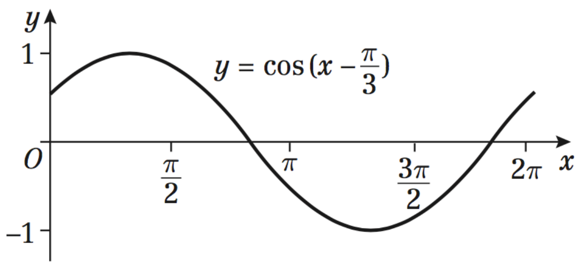
Crosses x-axis when $\sin\left(x + \frac{3\pi}{4}\right) = 0$

$$x + \frac{3\pi}{4} = -\pi, 0, \pi, 2\pi$$

$$x = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$$

So coordinates are $\left(0, \frac{1}{\sqrt{2}}\right), \left(-\frac{7\pi}{4}, 0\right), \left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right)$

2 a $y = \cos\left(x - \frac{\pi}{3}\right)$ is $y = \cos x$ translated by the vector $\begin{pmatrix} \frac{\pi}{3} \\ 0 \end{pmatrix}$



b Crosses y-axis when $y = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

Crosses x-axis when $\cos\left(x - \frac{\pi}{3}\right) = 0$

$$x - \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{5\pi}{6}, \frac{11\pi}{6}$$

So coordinates are $\left(0, \frac{1}{2}\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

2 c $\cos\left(x - \frac{\pi}{3}\right) = -0.27, \; 0 \leq x \leq 2\pi$

$\cos^{-1}(-0.27) = 1.844$ (3 d.p.)

$\Rightarrow x - \frac{\pi}{3} \approx 1.844$ and $x - \frac{\pi}{3} \approx 2\pi - 1.844$

$\Rightarrow x = 2.89, 5.49$ (2 d.p.)

3 a Let C be the midpoint of the line AB, then
AOC is a right-angled triangle and $AC = 3\,\text{cm}$, so

$$\sin\frac{\theta}{2} = \frac{3}{5} = 0.6 \Rightarrow \frac{\theta}{2} = 0.6435...$$

$\theta = 1.287$ radians (3 d.p.)

b Use $l = r\theta$
So arc $AB = 5\times 1.287 = 6.44\,\text{cm}$ (3 s.f.)

4 As ABC is equilateral, $BC = AC = 8\,\text{cm}$
$BP = AB - AP = 8 - 6 = 2\,\text{cm}$
$QC = BP = 2\,\text{cm}$

$\angle BAC = \frac{\pi}{3}$, $PQ = 6\times\frac{\pi}{3} = 2\pi = 6.28\,\text{cm}$ (2 d.p.)

So perimeter $= BC + BP + PQ + QC = 18.28\,\text{cm}$ (2 d.p.)

Exact answer $12 + 2\pi$ cm

5 a $\frac{1}{2}(r+10)^2\theta - \frac{1}{2}r^2\theta = 40$

$\Rightarrow 20r\theta + 100\theta = 80$

$\Rightarrow r\theta + 5\theta = 4$

$\Rightarrow r = \frac{4}{\theta} - 5$

b $r = \frac{4}{\theta} - 5 = 6\theta$

$\Rightarrow 4 - 5\theta = 6\theta^2$

$\Rightarrow 6\theta^2 + 5\theta - 4 = 0$

$\Rightarrow (3\theta + 4)(2\theta - 1) = 0$

$\Rightarrow \theta = -\frac{4}{3}$ or $\frac{1}{2}$

But θ cannot be negative, so $\theta = \frac{1}{2}$, $r = 3$

So perimeter $= 20 + r\theta + (10+r)\theta = 20 + \frac{3}{2} + \frac{13}{2} = 28\,\text{cm}$

6 a arc $BD = 10\times 0.6 = 6\,\text{cm}$

b Area of triangle $ABC = \frac{1}{2}(13\times 10)\sin 0.6 = 65\times 0.567 = 36.7\,\text{cm}^2$ (1 d.p.)

Area of sector $ABD = \frac{1}{2}10^2\times 0.6 = 30\,\text{cm}^2$

Area of shaded area $BCD = 36.7 - 30 = 6.7\,\text{cm}^2$ (1 d.p.)

7 a $\angle OED = 90^\circ$ because BC is parallel to ED

So $r = \dfrac{10}{\cos 0.7} = 13.07\,\text{cm}$ (2 d.p.)

Area of sector $OAB = \dfrac{1}{2}r^2 \times 1.4 = 119.7\,\text{cm}^2$ (1 d.p.)

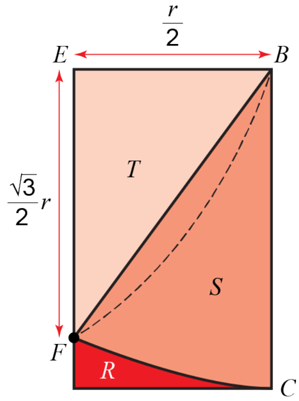
b $BC = AC = r\tan 0.7$

So perimeter $= 2r\tan 0.7 + r \times 1.4$

$= (2 \times 13.07 \times 0.842) + (13.07 \times 1.4) = 40.3\,\text{cm}$

8 Split each half of the rectangle as shown.

EFB is a right-angled triangle, and by Pythagoras' theorem $EF = \frac{\sqrt{3}}{2}r$

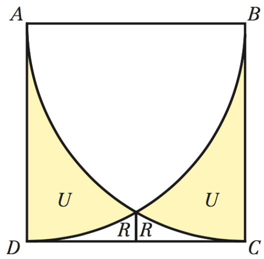


Let $\angle EBF = \theta$, so $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

So $\angle FBC = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

Area $S = \frac{1}{2}r^2\frac{\pi}{6} = \frac{\pi}{12}r^2$

Area $T = \frac{1}{2} \times \frac{\sqrt{3}}{2}r \times \frac{1}{2}r = \frac{\sqrt{3}}{8}r^2$



$\Rightarrow$ Area $R = \frac{1}{2}r^2 -$ Area $S -$ Area $T = \left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)r^2$

Area of sector $ACB = \frac{1}{2}r^2\frac{\pi}{2} = \frac{\pi}{4}r^2$

Area U = Area $ABCD -$ Area sector $ACB - 2R$

$$= r^2 - \frac{\pi}{4}r^2 - 2\left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)r^2$$

$$= r^2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right)$$

So area $U = r^2 - \frac{\pi}{4}r^2 - 2R$

$$= \left(1 - \frac{\pi}{4} - 1 + \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)r^2$$

$$= r^2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right) = \frac{r^2}{12}\left(3\sqrt{3} - \pi\right)$$

So shaded area $= 2U = \frac{r^2}{6}\left(3\sqrt{3} - \pi\right)$

9 a $3\sin^2 x + 7\cos x + 3 = 3(1-\cos^2 x) + 7\cos x + 3$

$= -3\cos^2 x + 7\cos x + 6$

$= 3\cos^2 x - 7\cos x - 6$

b $3\cos^2 x - 7\cos x - 6 = 0$

$(3\cos x + 2)(\cos x - 3) = 0$

$\cos x = -\frac{2}{3}$ or 3

$\cos x$ cannot be 3

so $\cos x = -\frac{2}{3}$

$x = 2.30, 2\pi - 2.30 = 2{,}30, 3.98$ (2 d.p.)

10 a For small values of θ:

$\sin 4\theta \approx 4\theta$

$\cos 4\theta \approx 1 - \frac{1}{2}(4\theta)^2 \approx 1 - 8\theta^2$,

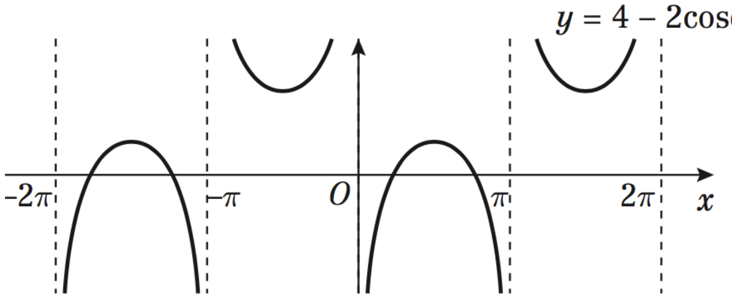
$\tan 3\theta \approx 3\theta$

$\sin 4\theta - \cos 4\theta + \tan 3\theta \approx 4\theta - (1 - 8\theta^2) + 3\theta$

$\approx 8\theta^2 + 7\theta - 1$

b -1

11 a $y = 4 - 2\operatorname{cosec} x$ is $y = \operatorname{cosec} x$ stretched by a scale factor 2 in the y-direction, then reflected in the x-axis and then translated by the vector $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$



b The minima in the graph occur when cosec $x = -1$ and $y = 6$. The maxima occur when cosec $x = 1$ and $y = 2$. So there are no solutions for $2 < k < 6$.

12 a The graph is a translation of $y = \sec\theta$ by α.

So $\alpha = \frac{\pi}{3}$

b As the curve passes through (0, 4)

$4 = k\sec\frac{\pi}{3} \Rightarrow k = 4\cos\frac{\pi}{3} = 2$

12 c $-2\sqrt{2} = 2\sec\left(\theta - \frac{\pi}{3}\right)$

$\Rightarrow \cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \theta - \frac{\pi}{3} = -\frac{5\pi}{4}, -\frac{3\pi}{4}$

$\Rightarrow \theta = -\frac{11\pi}{12}, -\frac{5\pi}{12}$

13 a $\frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} \equiv \frac{\cos^2 x + (1-\sin x)^2}{\cos x(1-\sin x)}$

$\equiv \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{\cos x(1-\sin x)}$

$\equiv \frac{2-2\sin x}{\cos x(1-\sin x)}$

$\equiv \frac{2}{\cos x}$

$\equiv 2\sec x$

b By part a the equation becomes

$2\sec x = -2\sqrt{2}$

$\Rightarrow \sec x = -\sqrt{2}$

$\Rightarrow \cos x = -\frac{1}{\sqrt{2}}$

$x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$

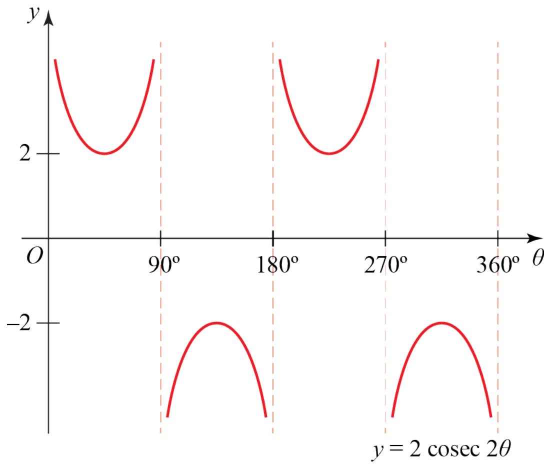
14 a $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\sin\theta\cos\theta}$ (using $\cos^2\theta + \sin^2\theta \equiv 1$)

$= \frac{1}{\frac{1}{2}\sin 2\theta}$ (using double-angle formula $\sin 2\theta \equiv 2\sin\theta\cos\theta$)

$= 2\operatorname{cosec} 2\theta$

14b The graph of $y = 2\operatorname{cosec} 2\theta$ is a stretch of the graph of $y = \operatorname{cosec}\theta$ by a scale factor of $\frac{1}{2}$ in the horizontal direction and then a stretch by a factor of 2 in the vertical direction.



c By part a the equation becomes

$2\operatorname{cosec} 2\theta = 3$

$\Rightarrow \operatorname{cosec} 2\theta = \dfrac{3}{2}$

$\Rightarrow \sin 2\theta = \dfrac{2}{3}$, in the interval $0 \le 2\theta \le 720°$

Calculator value is $2\theta = 41.81°$ (2 d.p.)

Solutions are $2\theta = 41.81°, 180° - 41.81°, 360° + 41.81°, 540° - 41.81°$

So the solution set is: 20.9°, 69.1°, 200.9°, 249.1°

15 a Note the angle $BDC = \theta$

$\cos\theta = \dfrac{BC}{10} \Rightarrow BC = 10\cos\theta$

$\sin\theta = \dfrac{BC}{BD} \Rightarrow BD = \dfrac{BC}{\sin\theta} = \dfrac{10\cos\theta}{\sin\theta} = 10\cot\theta$

b $10\cot\theta = \dfrac{10}{\sqrt{3}}$

$\Rightarrow \cot\theta = \dfrac{1}{\sqrt{3}},\ \ \theta = \dfrac{\pi}{3}$

From the triangle BCD, $\cos\theta = \dfrac{DC}{BD}$

$\Rightarrow DC = BD\cos\theta$

So $DC = 10\cot\theta\cos\theta$

$= 10\left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{1}{2}\right)$

$= \dfrac{5}{\sqrt{3}}$

16 a $\sin^2\theta + \cos^2\theta \equiv 1$

$\Rightarrow \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} \equiv \dfrac{1}{\cos^2\theta}$ (dividing by $\cos^2\theta$)

$\Rightarrow \tan^2\theta + 1 \equiv \sec^2\theta$

b $2\tan^2\theta + \sec\theta = 1$

$\Rightarrow 2\sec^2\theta - 2 + \sec\theta = 1$

$\Rightarrow 2\sec^2\theta + \sec\theta - 3 = 0$

$\Rightarrow (2\sec\theta + 3)(\sec\theta - 1) = 0$

$\Rightarrow \sec\theta = -\dfrac{3}{2}, \sec\theta = 1$

$\Rightarrow \cos\theta = -\dfrac{2}{3}, \cos\theta = 1$

Solutions are $131.8°, 360° - 131.8°, 0°$

So solution set is: $0.0°, 131.8°, 228.2°$ (1 d.p.)

17 a $a = \dfrac{1}{\sin x} = \dfrac{1}{\frac{1}{2}b} = \dfrac{2}{b}$

b $\dfrac{4-b^2}{a^2-1} = \dfrac{4-b^2}{\left(\frac{2}{b}\right)^2 - 1}$

$= \dfrac{4-b^2}{\frac{4}{b^2} - 1} = \dfrac{4-b^2}{\frac{4-b^2}{b^2}} = (4-b^2) \times \dfrac{b^2}{4-b^2}$

$= b^2$

An alternative approach is to first substitute the trigonometric functions for a and b

$\dfrac{4-b^2}{a^2-1} = \dfrac{4-4\sin^2 x}{\operatorname{cosec}^2 x - 1}$

$= \dfrac{4(1-\sin^2 x)}{\cot^2 x}$

$= \dfrac{4\cos^2 x}{\cot^2 x}$

$= 4\sin^2 x = b^2$

18 a $y = \arcsin x$

$\Rightarrow \sin y = x$

$x = \cos\left(\frac{\pi}{2} - y\right)$ ← Using $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \frac{\pi}{2} - y = \arccos x$

b $\arcsin x + \arccos x = y + \frac{\pi}{2} - y$

$= \frac{\pi}{2}$

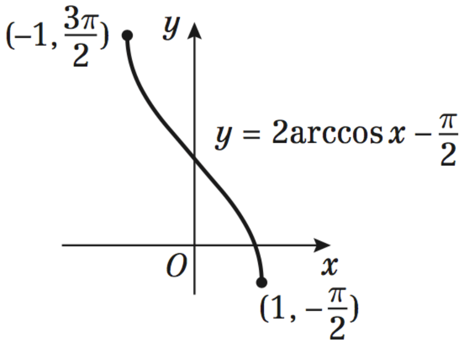
19 a $\arccos\frac{1}{x} = p \Rightarrow \cos p = \frac{1}{x}$

Use Pythagoras' theorem to show that opposite side of the right-angle triangle with angle p is $\sqrt{x^2-1}$

So $\sin p = \frac{\sqrt{x^2-1}}{x} \Rightarrow p = \arcsin\frac{\sqrt{x^2-1}}{x}$

b If $0 \le x < 1$ then $x^2 - 1$ is negative and you cannot take the square root of a negative number.

20 a $y = 2\arccos x - \frac{\pi}{2}$ is $y = \arccos x$ stretched by a scale factor of 2 in the y-direction and then translated by $-\frac{\pi}{2}$ in the vertical direction



b $2\arccos x - \frac{\pi}{2} = 0$

$\Rightarrow \arccos x = \frac{\pi}{4}$

$\Rightarrow x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Coordinates are $\left(\frac{1}{\sqrt{2}}, 0\right)$

21 $\tan\left(x+\frac{\pi}{6}\right) = \frac{1}{6} \Rightarrow \frac{\tan x + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}\tan x} = \frac{1}{6}$ [using the addition formula for tan $(A + B)$]

$6\tan x + 2\sqrt{3} = 1 - \frac{\sqrt{3}}{3}\tan x$

$\left(\frac{18+\sqrt{3}}{3}\right)\tan x = 1 - 2\sqrt{3}$

$\tan x = \frac{3\left(1-2\sqrt{3}\right)\left(18-\sqrt{3}\right)}{\left(18+\sqrt{3}\right)\left(18-\sqrt{3}\right)}$

$= \frac{72-111\sqrt{3}}{321}$

22 a $\sin(x+30°) = 2\sin(x+60°)$

So $\sin x\cos 30° + \cos x\sin 30° = 2(\sin x\cos 60° - \cos x\sin 60°)$ (using the addition formulae for sin)

$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)$

$\sqrt{3}\sin x + \cos x = 2\sin x - 2\sqrt{3}\cos x$ (multiplying both sides by 2)

$\left(-2+\sqrt{3}\right)\sin x = \left(-1-2\sqrt{3}\right)\cos x$

So $\tan x = \frac{-1-2\sqrt{3}}{-2+\sqrt{3}}$

$= \frac{\left(-1-2\sqrt{3}\right)\left(-2-\sqrt{3}\right)}{\left(-2+\sqrt{3}\right)\left(-2-\sqrt{3}\right)}$

$= \frac{2+6+4\sqrt{3}+\sqrt{3}}{4-3}$

$= 8+5\sqrt{3}$

b $\tan(x+60°) = \frac{\tan x + \tan 60}{1-\tan x\tan 60}$

$= \frac{8+5\sqrt{3}+\sqrt{3}}{1-\left(8+5\sqrt{3}\right)\sqrt{3}}$

$= \frac{8+6\sqrt{3}}{-14-8\sqrt{3}}$

$= \frac{\left(4+3\sqrt{3}\right)\left(-7+4\sqrt{3}\right)}{\left(-7-4\sqrt{3}\right)\left(-7+4\sqrt{3}\right)}$

$= \frac{36-28-21\sqrt{3}+16\sqrt{3}}{49-48}$

$= 8-5\sqrt{3}$

23 a $\sin 165° = \sin(120°+45°)$

$= \sin 120°\cos 45° + \cos 120°\sin 45°$

$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} + \frac{-1}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

23 b $\text{cosec}165° = \dfrac{1}{\sin 165°}$

$= \dfrac{4}{\left(\sqrt{6}-\sqrt{2}\right)} \times \dfrac{\left(\sqrt{6}+\sqrt{2}\right)}{\left(\sqrt{6}+\sqrt{2}\right)}$

$= \dfrac{4\left(\sqrt{6}+\sqrt{2}\right)}{6-2}$

$= \sqrt{6}+\sqrt{2}$

24 a $\cos A = \dfrac{3}{4}$

Using Pythagoras' theorem and noting that sin A is negative as A is in the fourth quadrant, this gives

$\sin A = -\dfrac{\sqrt{7}}{4}$

Using the double-angle formula for sin gives

$\sin 2A = 2\sin A\cos A = 2\left(-\dfrac{\sqrt{7}}{4}\right)\left(\dfrac{3}{4}\right) = -\dfrac{3\sqrt{7}}{8}$

b $\cos 2A = 2\cos^2 A - 1 = \dfrac{1}{8}$

$\Rightarrow \tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{\left(-\frac{3\sqrt{7}}{8}\right)}{\left(\frac{1}{8}\right)} = -3\sqrt{7}$

25 a $\cos 2x + \sin x = 1$

$\Rightarrow 1 - 2\sin^2 x + \sin x = 1$ (using double-angle formula for $\cos 2x$)

$\Rightarrow 2\sin^2 x - \sin x = 0$

$\Rightarrow \sin x(2\sin x - 1) = 0$

$\Rightarrow \sin x = 0, \sin x = \dfrac{1}{2}$

Solutions in the given interval are: $-180°, 0°, 30°, 150°, 180°$

b $\sin x(\cos x + \text{cosec}\, x) = 2\cos^2 x$

$\Rightarrow \sin x \cos x + 1 = 2\cos^2 x$

$\Rightarrow \text{in}\, x \cos x = 2\cos^2 x - 1$

$\Rightarrow \dfrac{1}{2}\sin 2x = \cos 2x$ (using the double-angle formulae for $\sin 2x$ and $\cos 2x$)

$\Rightarrow \tan 2x = 2,$ for $-360° \leq 2x \leq 360°$

So $2x = 63.43° - 360°, 63.43° - 180°, 63.43°, 63.43° + 180°$

Solution set: $-148.3°, -58.3°, 31.7°, 121.7°$ (1 d.p.)

26 a $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

So $R\cos\alpha = 3$, $R\sin\alpha = 2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 2^2 = 9 + 4 = 13$

$\Rightarrow R = \sqrt{13}$ (as $\cos^2\alpha + \sin^2\alpha \equiv 1$)

$\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 0.588$ (3 d.p.)

b $R^4 = \left(\sqrt{13}\right)^4 = 169$ since the maximum value the sin function can take is 1

c $\sqrt{13}\sin(x+0.588) = 1$

$\sin(x+0.5880) = \frac{1}{\sqrt{13}} = 0.27735\ldots$

$x + 0.588 = \pi - 0.281, 2\pi + 0.281$

$x = 2.273, 5.976$ (3 d.p.)

27 a LHS $\equiv \cot\theta - \tan\theta$

$\equiv \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$

$\equiv \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$

$\equiv \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta}$ (using the double angle formulae for $\sin 2\theta$ and $\cos 2\theta$)

$\equiv 2\cot 2\theta \equiv$ RHS

b $2\cot 2\theta = 5 \Rightarrow \cot 2\theta = \frac{5}{2} \Rightarrow \tan 2\theta = \frac{2}{5}$, for $-2\pi < 2\theta < 2\pi$

So $2\theta = 0.3805 - 2\pi, 0.3805 - \pi, 0.3805, 0.3805 + \pi$

Solution set: $-2.95, -1.38, 0.190, 1.76$ (3 s.f.)

28 a LHS $\equiv \cos 3\theta$

$\equiv \cos(2\theta + \theta)$

$\equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - \left(2\sin\theta\cos\theta\right)\sin\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta \equiv$ RHS

b From part a $\cos 3\theta = 4\frac{2\sqrt{2}}{27} - \sqrt{2} = -\frac{19\sqrt{2}}{27}$

So $\sec 3\theta = -\frac{27}{19\sqrt{2}} = -\frac{27\sqrt{2}}{38}$

29 $\sin^4\theta = (\sin^2\theta)(\sin^2\theta)$

Use the double-angle formula to write $\sin^2\theta$ in terms of $\cos 2\theta$

$\cos 2\theta = 1 - \sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

Now substitute the expression for $\sin^2\theta$ and expand the brackets

$$\text{So } \sin^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= \frac{1}{4}\left(1 - 2\cos 2\theta + \cos^2 2\theta\right)$$

Again use the double-angle formula to write $\cos^2 2\theta$ in terms of $\cos 4\theta$

$$\text{So } \sin^4\theta = \frac{1}{4}\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$
$$= \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

30 a $R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha = 6\sin\theta + 2\cos\theta$

So $R\cos\alpha = 6$, $R\sin\alpha = 2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 6^2 + 2^2 = 36 + 4 = 40$

$\Rightarrow R = \sqrt{40}$ (as $\cos^2\alpha + \sin^2\alpha \equiv 1$)

$\tan\alpha = \dfrac{2}{6} \Rightarrow \alpha = 0.32175\ldots = 0.32$ (2 d.p.)

So $6\sin\theta + 2\cos\theta \approx \sqrt{40}\sin(\theta + 0.32)$

b i $\sqrt{40}$, since the maximum value the sin function can take is 1

ii Maximum occurs when $\sin(\theta + 0.322) = 1$

$\Rightarrow \theta + 0.322 = \dfrac{\pi}{2}$

$\Rightarrow \theta = 1.25$ (2 d.p.)

Note that you should use a value of α to 3 decimal places in the model and then give your answers to 2 decimal places.

c $T = 9 + \sqrt{40}\sin\left(\dfrac{\pi t}{12} + 0.322\right)$

So minimum value of T is $9 - \sqrt{40} = 2.68\,°\text{C}$ (2 d.p.)

Occurs when $\sin\left(\dfrac{\pi t}{12} + 0.322\right) = -1$

$\Rightarrow \dfrac{\pi t}{12} + 0.322 = \dfrac{3\pi}{2}$

$\Rightarrow t = 16.77$ hours

30 d $9+\sqrt{40}\sin\left(\frac{\pi t}{12}+0.322\right)=14$

$\Rightarrow \sqrt{40}\sin\left(\frac{\pi t}{12}+0.322\right)=5$

$\Rightarrow \sin\left(\frac{\pi t}{12}+0.322\right)=\frac{5}{\sqrt{40}}$

$\Rightarrow \frac{\pi t}{12}+0.322=0.9117, 2.2299$

$\Rightarrow t=2.25, 7.29$ (2 d.p.)

$0.25\,\text{h}\approx 15$ minutes and $0.29\,\text{h}\approx 17$ minutes

So times are 11:15 am and 4:17 pm

31 a As $\frac{4}{t}\neq 0, x\neq 1$

The equation for y can be rewritten as

$y=\left(t-\frac{3}{2}\right)^2-\frac{5}{4}$

So $y\geq -1.25$

b $t=\frac{4}{1-x}$

So $y=\left(\frac{4}{1-x}\right)^2-3\left(\frac{4}{1-x}\right)+1$

$=\frac{16}{(1-x)^2}-\frac{12(1-x)}{(1-x)^2}+\frac{(1-x)^2}{(1-x)^2}$

$=\frac{16-12+12x+1-2x+x^2}{(1-x)^2}$

$=\frac{x^2+10x+5}{(1-x)^2}$

So $a=1,\ b=10,\ c=5$

32 a $x=\ln(t+2)\Rightarrow e^x=t+2\Rightarrow t=e^x-2$

$y=\frac{3t}{t+3}=\frac{3e^x-6}{e^x+1}$

$t>4\Rightarrow e^x-2>4\Rightarrow e^x>6\Rightarrow x>\ln 6$

So the solution is $y=\frac{3e^x-6}{e^x+1}, x>\ln 6$

32 b When $x \to \infty$, $y \to 3$

When $x = \ln 6$, $y = \dfrac{3e^{\ln 6} - 6}{e^{\ln 6} + 1} = \dfrac{(3 \times 6) - 6}{6 + 1} = \dfrac{12}{7}$

So range is $\dfrac{12}{7} < y < 3$

33 $x = \dfrac{1}{1+t} \Rightarrow t = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

$y = \dfrac{1}{1 - \dfrac{1-x}{x}} = \dfrac{x}{x - (1-x)} = \dfrac{x}{2x-1}$

34 a $y = \cos 3t = \cos(2t + t) = \cos 2t \cos t - \sin 2t \sin t$

$= (\cos^2 t - 1)\cos t - 2\sin^2 t \cos t$

$= 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t$

$= 4\cos^3 t - 3\cos t$

$x = 2\cos t \Rightarrow \cos t = \dfrac{x}{2}$

$y = 4\left(\dfrac{x}{2}\right)^3 - 3\left(\dfrac{x}{2}\right) = \dfrac{1}{2}x^3 - \dfrac{3}{2}x = \dfrac{x}{2}(x^2 - 3)$

b $0 \le t \le \dfrac{\pi}{2}$

So $0 \le \cos t \le 1$ and $-1 \le \cos 3t \le 1$

So $0 \le x \le 2, -1 \le y \le 1$

35 a $y = \sin\left(t + \dfrac{\pi}{6}\right) = \sin t \cos \dfrac{\pi}{6} + \cos t \sin \dfrac{\pi}{6}$

$= \dfrac{\sqrt{3}}{2}\sin t + \dfrac{1}{2}\cos t$

$= \dfrac{\sqrt{3}}{2}\sin t + \dfrac{1}{2}\sqrt{1 - \sin^2 t}$

$= \dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}\sqrt{1 - x^2}$

As $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}, -1 \le \sin t \le 1 \Rightarrow -1 \le x \le 1$

35 b At A, $\sin\left(t+\frac{\pi}{6}\right)=0 \Rightarrow t=-\frac{\pi}{6}$

$x=\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

Coordinates of A are $\left(-\frac{1}{2}, 0\right)$

At B, $x=\sin t=0 \Rightarrow t=0$

$y=\sin\left(t+\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

Coordinates of B are $\left(0, \frac{1}{2}\right)$

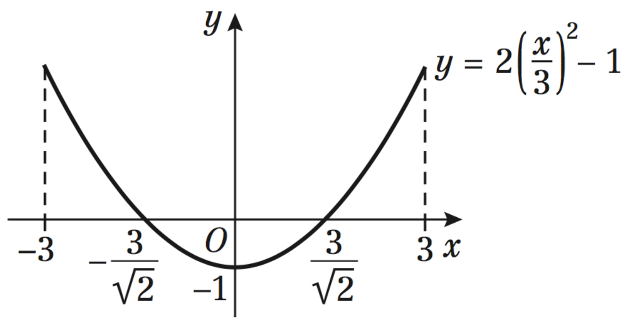
36 a $y=\cos 2t=2\cos^2 t-1$

$y=2\left(\frac{x}{3}\right)^2-1,\ -3\leq x\leq 3$

b Curve is a parabola, with a minima and y-intercept at (0,–1) and x-intercepts when

$2\left(\frac{x}{3}\right)^2=1 \Rightarrow \frac{x}{3}=\pm\frac{1}{\sqrt{2}} \Rightarrow x=\pm\frac{3}{\sqrt{2}}$

Coordinates $\left(-\frac{3}{\sqrt{2}}, 0\right), \left(\frac{3}{\sqrt{2}}, 0\right)$



37 $y=3x+c$ would intersect curve C if

$8t(2t-1)=3(4t)+c$

$16t^2-20t-c=0$

Using the quadratic formula, this equation has no real solutions if

$(-20)^2-4(16)(-c)<0$

$\Rightarrow\ 64c<-400 \Rightarrow c<-\frac{25}{4}$

38 a The curve intersects the x-axis when $2\cos t + 1 = 0 \Rightarrow \cos t = -\frac{1}{2}$

Solutions in the interval are $t = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\Rightarrow x = 3\sin\left(\frac{4\pi}{3}\right), 3\sin\left(\frac{8\pi}{3}\right)$

So coordinates are $\left(-\frac{3\sqrt{3}}{2}, 0\right)$ and $\left(\frac{3\sqrt{3}}{2}, 0\right)$

b $3\sin 2t = 1.5 \Rightarrow \sin 2t = \frac{1}{2}$

In the interval $\pi \leq 2t \leq 3\pi$ solutions are

$2t = \frac{13\pi}{6}, \frac{17\pi}{6}$

$\Rightarrow t = \frac{13\pi}{12}, \frac{17\pi}{12}$

39 a Find the time the ball hits the ground by solving $-4.9t^2 + 25t + 50 = 0$

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(50)}}{2(-4.9)}$$

$t \geq 0$, so only valid solution is $t = 6.64$ s (2 d.p.)

$\Rightarrow k = 6.64$ (2 d.p.)

b $t = \frac{x}{25\sqrt{3}}$

$$y = 25\left(\frac{x}{25\sqrt{3}}\right) - 4.9\left(\frac{x}{25\sqrt{3}}\right)^2 + 50$$

$$= \frac{x}{\sqrt{3}} - \frac{49}{18750}x^2 + 50$$

Domain of the function is from where the ball is hit at $x = 0$ to where it hits the ground when $t = 6.64$ seconds.

When $t = 6.64$, $x = 25\sqrt{3}(6.64) = 287.5$ (1 d.p.)

So domain is $0 \leq x \leq 287.5$

Challenge

1 Angle of minor arc = $\frac{\pi}{2}$ because it is a quarter circle

Let the chord meet the circle at R and T. The area of P is the area of sector formed by O, R and T less the area of the triangle ORT.

So area of P = $\frac{1}{2}r^2\frac{\pi}{2}-\frac{1}{2}r^2\sin\frac{\pi}{2}=r^2\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{r^2}{4}(\pi-2)$

Area of Q = $\pi r^2 -$ area of P

$$=r^2\left(\pi-\frac{\pi}{4}+\frac{1}{2}\right)=r^2\left(\frac{3\pi}{4}+\frac{1}{2}\right)=\frac{r^2}{4}(3\pi+2)$$

So ratio = $(\pi-2):(3\pi+2)=\frac{\pi-2}{3\pi+2}:1$

2 **a** $\sin x$

b $\cos x$

c $\angle COA=\frac{\pi}{2}-x\Rightarrow\angle CAO=x$

$OA=1\div\sin x=\text{cosec}\,x$

d $AC=1\div\tan x=\cot x$

e $\tan x$

f $OB=1\div\cos x=\sec x$

3 a $\sin t = \dfrac{x-3}{4}$, $\cos t = \dfrac{y+1}{4}$

As $\sin^2 t + \cos^2 t = 1$

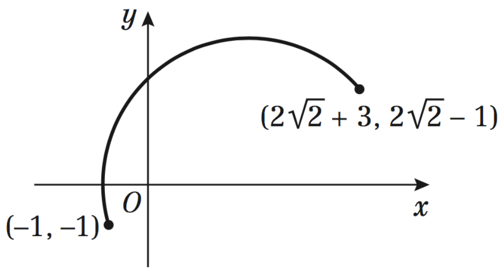
$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{4}\right)^2 = 1$$

$$\Rightarrow (x-3)^2 + (y+1)^2 = 16$$

The curve is a circle centre (3, –1) and radius 4.

Endpoints when $t = -\dfrac{\pi}{2}$, $x = -1$, $y = -1$

and when $t = \dfrac{\pi}{4}$, $x = 2\sqrt{2} + 3$, $y = 2\sqrt{2} - 1$



b C is $\frac{3}{8}$ths of a circle, radius 4

So length = $\dfrac{3}{8} \times 8\pi = 3\pi$