Review exercise 3

1 $y = \frac{1}{2}x^2 + 4\cos x$

$\frac{dy}{dx} = x - 4\sin x$

When $x = \frac{\pi}{2}$:

$y = \frac{\pi^2}{8}$ and $\frac{dy}{dx} = \frac{\pi}{2} - 4 = \frac{\pi - 8}{2}$

So gradient of normal is $-\frac{2}{\pi - 8}$

Equation of normal is

$y - \frac{\pi^2}{8} = -\frac{2}{\pi - 8}\left(x - \frac{\pi}{2}\right)$

$y(8-\pi) - \frac{\pi^2}{8}(8-\pi) = 2\left(x - \frac{\pi}{2}\right)$

$8y(8-\pi) - \pi^2(8-\pi) = 16x - 8\pi$

$8y(8-\pi) - 16x - \pi^2(8-\pi) + 8\pi = 0$

$8y(8-\pi) - 16x + \pi(\pi^2 - 8\pi + 8) = 0$

2 $y = e^{3x} - \ln(x^2)$

$= e^{3x} - 2\ln x$

$\frac{dy}{dx} = 3e^{3x} - \frac{2}{x}$

When $x = 2$:

$y = e^6 - \ln 4$ and $\frac{dy}{dx} = 3e^6 - 1$

Equation of tangent is

$y - (e^6 - \ln 4) = (3e^6 - 1)(x - 2)$

$y - e^6 + \ln 4 = (3e^6 - 1)x - 6e^6 + 2$

$y - (3e^6 - 1)x - 2 + \ln 4 + 5e^6 = 0$

3 $y = \frac{3}{(4-6x)^2}$

$= 3(4-6x)^{-2}$

$\frac{dy}{dx} = 36(4-6x)^{-3} = \frac{36}{(4-6x)^3}$

When $x = 1$:

$y = \frac{3}{4}$ and $\frac{dy}{dx} = -\frac{36}{8} = -\frac{9}{2}$

So gradient of normal is $\frac{2}{9}$

Equation of normal is

$y - \frac{3}{4} = \frac{2}{9}(x-1)$

$36y - 27 = 8x - 8$

$0 = 8x - 36y + 19$

4 a $y = (2x-3)^2 e^{2x}$

Let $u = (2x-3)^2 \Rightarrow \frac{du}{dx} = 4(2x-3)$

and $v = e^{2x} \Rightarrow \frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= 2(2x-3)^2 e^{2x} + 4(2x-3)e^{2x}$

$= 2e^{2x}(2x-3)(2x-3+2)$

$= 2e^{2x}(2x-3)(2x-1)$

b $\frac{dy}{dx} = 0 \Rightarrow 2x - 3 = 0$ or $2x - 1 = 0$

So $x = \frac{3}{2}$ or $\frac{1}{2}$

When $x = \frac{3}{2}$, $y = 0$

When $x = \frac{1}{2}$, $y = 4e$

So coordinates of stationary points are $\left(\frac{3}{2}, 0\right)$ and $\left(\frac{1}{2}, 4e\right)$.

5 a $y = \dfrac{(x-1)^2}{\sin x}$

Let $u = (x-1)^2 \Rightarrow \dfrac{du}{dx} = 2(x-1)$

and $v = \sin x \Rightarrow \dfrac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{2(x-1)\sin x - (x-1)^2 \cos x}{\sin^2 x}$$
$$= \frac{(x-1)(2\sin x - x\cos x + \cos x)}{\sin^2 x}$$

b When $x = \dfrac{\pi}{2}$:

$y = \left(\dfrac{\pi}{2} - 1\right)^2$ and $\dfrac{dy}{dx} = 2\left(\dfrac{\pi}{2} - 1\right)$

Equation of tangent is

$$y - \left(\frac{\pi}{2} - 1\right)^2 = 2\left(\frac{\pi}{2} - 1\right)\left(x - \frac{\pi}{2}\right)$$
$$= (\pi - 2)\left(x - \frac{\pi}{2}\right)$$
$$y = (\pi - 2)x - \frac{\pi}{2}(\pi - 2) + \left(\frac{\pi}{2} - 1\right)^2$$
$$= (\pi - 2)x - \frac{\pi^2}{2} + \pi + \frac{\pi^2}{4} - \pi + 1$$
$$= (\pi - 2)x + \left(1 - \frac{\pi^2}{4}\right)$$

6 a $y = \operatorname{cosec} x = \dfrac{1}{\sin x}$

Let $u = \sin x \Rightarrow \dfrac{du}{dx} = \cos x$

and $y = \dfrac{1}{u} \Rightarrow \dfrac{dy}{du} = -\dfrac{1}{u^2}$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -\frac{1}{\sin^2 x} \times \cos x$$
$$= -\frac{1}{\sin x} \times \frac{1}{\tan x}$$
$$= -\operatorname{cosec} x \cot x$$

b $x = \operatorname{cosec} 6y$

$$\frac{dx}{dy} = -6\operatorname{cosec} 6y \cot 6y$$

$\operatorname{cosec}^2 6y = 1 + \cot^2 6y$

$\Rightarrow \cot 6y = \sqrt{x^2 - 1}$

$$\frac{dx}{dy} = -6x\sqrt{x^2 - 1}$$
$$\frac{dy}{dx} = -\frac{1}{6x\sqrt{x^2 - 1}}$$

7 $y = \arcsin x$

So $x = \sin y$

$\Rightarrow \dfrac{dx}{dy} = \cos y$ and $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$\sin^2 y + \cos^2 y = 1$

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

So $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$

8 a $x = 2\cot t, y = \sin^2 t$

$$\frac{dx}{dt} = -2\operatorname{cosec}^2 t, \quad \frac{dy}{dt} = 4\sin t\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= \frac{4\sin t\cos t}{-2\operatorname{cosec}^2 t}$$
$$= -2\sin^3 t\cos t$$

b When $t = \frac{\pi}{4}$:

$$x = 2 \text{ and } y = 2\times\left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$\frac{dy}{dx} = -2\times\left(\frac{1}{\sqrt{2}}\right)^3\times\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$$

So equation of tangent is

$$y - 1 = -\frac{1}{2}(x-2)$$
$$y = -\frac{1}{2}x + 2$$

c $x = 2\cot t \Rightarrow \cot t = \frac{x}{2}$

$y = 2\sin^2 t \Rightarrow \sin^2 t = \frac{y}{2}$ and $\operatorname{cosec}^2 t = \frac{2}{y}$

$$\operatorname{cosec}^2 t = 1 + \cot^2 t$$
$$\frac{2}{y} = 1 + \left(\frac{x}{2}\right)^2$$
$$= \frac{4+x^2}{4}$$
$$\frac{y}{2} = \frac{4}{4+x^2}$$
$$y = \frac{8}{4+x^2}$$

As $0 < t \le \frac{\pi}{2}, \cot t \ge 0$

$x = 2\cot t$ so the domain of the function is $x \ge 0$.

9 a $x = \frac{1}{1+t}, y = \frac{1}{1-t}$

Using the chain rule:

$$\frac{dx}{dt} = \frac{-1}{(1+t)^2}, \quad \frac{dy}{dt} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= -\frac{(1+t)^2}{(1-t)^2}$$

When $t = \frac{1}{2}$:

$$x = \frac{2}{3} \text{ and } y = 2$$

$$\frac{dy}{dx} = -\frac{\left(\frac{3}{2}\right)^2}{\left(\frac{1}{2}\right)^2} = -\frac{\frac{9}{4}}{\frac{1}{4}} = -9$$

So equation of tangent is

$$y - 2 = -9\left(x - \frac{2}{3}\right)$$
$$y = -9x + 8$$

b $x = \frac{1}{1+t} \Rightarrow t = \frac{1}{x} - 1$

Substitute into $y = \frac{1}{1-t}$:

$$y = \frac{1}{1 - \left(\frac{1}{x} - 1\right)}$$
$$= \frac{1}{2 - \frac{1}{x}}$$
$$= \frac{x}{2x-1}$$

10 $3x^2 - 2y^2 + 2x - 3y + 5 = 0$

Differentiating with respect to x:

$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} + 0 = 0$

Substituting $x = 0,\ y = 1$:

$-4\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$

$7\frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{2}{7}$

So gradient of normal at (0, 1) is $\frac{-7}{2}$

Equation of normal is

$y - 1 = \frac{-7}{2}(x - 0)$

$y = \frac{-7}{2}x + 1$

$7x + 2y - 2 = 0$

11 a $\sin x + \cos y = 0.5$

Differentiating with respect to x:

$\cos x - \sin y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\sin y}$

b $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}$

When $x = \frac{\pi}{2}$:

$1 + \cos y = 0.5 \Rightarrow \cos y = -0.5$

$y = \frac{2\pi}{3}$ or $y = \frac{-2\pi}{3}$

When $x = -\frac{\pi}{2}$:

$-1 + \cos y = 0.5 \Rightarrow \cos y = 1.5$

(no solutions)

So the only stationary points in the given range are at $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, \frac{-2\pi}{3}\right)$.

12 $y = x^2 e^{-x}$

$\frac{dy}{dx} = -x^2 e^{-x} + 2x e^{-x} = e^{-x}\left(2x - x^2\right)$

$\frac{d^2y}{dx^2} = e^{-x}\left(2 - 2x\right) - e^{-x}\left(2x - x^2\right)$

$= e^{-x}\left(x^2 - 4x + 2\right)$

For C to be convex, $\frac{d^2y}{dx^2} \geq 0$.

$e^{-x} > 0$, and for all $x < 0$, $x^2 - 4x + 2 > 0$

So $\frac{d^2y}{dx^2} > 0$ for all $x < 0$.

Hence C is convex for all $x < 0$.

13 a $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

b Using the chain rule:

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{1000}{(2t+1)^2} = 4\pi r^2 \times \frac{dr}{dt}$

So $\frac{dr}{dt} = \frac{1000}{4\pi(2t+1)^2 r^2}$

$= \frac{250}{\pi(2t+1)^2 r^2}$

14 a $g(x) = x^3 - x^2 - 1$

$g(1.4) = 1.4^3 - 1.4^2 - 1 = -0.216 < 0$

$g(1.5) = 1.5^3 - 1.5^2 - 1 = 0.125 > 0$

The change of sign implies that the root α is in $[1.4, 1.5]$.

b $g(1.4655) = -0.00025... < 0$

$g(1.4665) = 0.00326... > 0$

The change of sign implies that the root α satisfies $1.4655 < \alpha < 1.4665$, and so $\alpha = 1.466$ correct to 3 decimal places.

15 a $p(x) = \cos x + e^{-x}$

$p(1.7) = \cos 1.7 + e^{-1.7} = 0.054... > 0$
$p(1.8) = \cos 1.8 + e^{-1.8} = -0.062... < 0$

The change of sign implies that the root α is in $[1.7, 1.8]$.

b $p(1.7455) = \cos 1.7455 + e^{-1.7455}$
$= 0.00074... > 0$
$p(1.7465) = \cos 1.7465 + e^{-1.7465}$
$= -0.00042... < 0$

The change of sign implies that the root α satisfies $1.7455 < \alpha < 1.7465$, and so $\alpha = 1.746$ correct to 3 decimal places.

16 a $f(x) = e^{x-2} - 3x + 5 = 0$
$e^{x-2} = 3x - 5$
$x - 2 = \ln(3x - 5)$
$x = \ln(3x-5) + 2$, for $3x - 5 > 0 \Rightarrow x > \frac{5}{3}$

b Using $x_0 = 4$:
$x_1 = \ln 7 + 2 = 3.9459$
$x_2 = \ln(3 \times 3.9459 - 5) + 2 = 3.9225$
$x_3 = \ln(3 \times 3.9225 - 5) + 2 = 3.9121$
All correct to 4 decimal places.

17 a $f(x) = \frac{1}{(x-2)^3} + 4x^2$

$f(0.2) = \frac{1}{(0.2-2)^3} + 4 \times 0.2^2$
$= -0.011... < 0$

$f(0.3) = \frac{1}{(0.3-2)^3} + 4 \times 0.3^2$
$= 0.156... > 0$

The change of sign implies that the root α is in $[0.2, 0.3]$.

b $f(x) = \frac{1}{(x-2)^3} + 4x^2 = 0$

$\frac{1}{(x-2)^3} = -4x^2$

$(x-2)^3 = -\frac{1}{4x^2}$

$x - 2 = \sqrt[3]{\frac{-1}{4x^2}}$

$x = \sqrt[3]{\frac{-1}{4x^2}} + 2$

c Using $x_0 = 1$:

$x_1 = \sqrt[3]{\frac{-1}{4}} + 2 = 1.3700$

$x_2 = \sqrt[3]{\frac{-1}{4 \times 1.3700^2}} + 2 = 1.4893$

$x_3 = \sqrt[3]{\frac{-1}{4 \times 1.4893^2}} + 2 = 1.5170$

$x_4 = \sqrt[3]{\frac{-1}{4 \times 1.5170^2}} + 2 = 1.5228$

All correct to 4 decimal places.

d $f(1.5235) = \frac{1}{(1.5235-2)^3} + 4 \times 1.5235^2$
$= 0.0412... > 0$

$f(1.5245) = \frac{1}{(1.5245-2)^3} + 4 \times 1.5245^2$
$= -0.0050... < 0$

The change of sign implies that the root α satisfies $1.5235 < \alpha < 1.5245$, and so $\alpha = 1.524$ correct to 3 decimal places.

18 a $f(x) = \frac{1}{10} x^2 e^x - 2x - 10$

As A is a stationary point, the gradient at A is zero. So $f'(a) = 0$.

The Newton–Raphson process uses $f'(x)$ as a denominator. Division by zero is undefined so $x_0 = a$ cannot be used to find an approximation for α.

18 b $f'(x) = xe^x(0.1x + 0.2) - 2$

Using $x_0 = 2.9$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2.9 - \frac{f(2.9)}{f'(2.9)}$$
$$= 2.9 + \frac{0.5155}{23.825}$$
$$= 2.922 \text{ (3 d.p.)}$$

19 a $f(x) = \frac{3}{10}x^3 - x^{\frac{2}{3}} + \frac{1}{x} - 4$

i $f(0.2) = \frac{3}{10} \times 0.2^3 - 0.2^{\frac{2}{3}} + \frac{1}{0.2} - 4$
$= 0.660... > 0$

$f(0.3) = \frac{3}{10} \times 0.3^3 - 0.3^{\frac{2}{3}} + \frac{1}{0.3} - 4$
$= -1.107... < 0$

The change of sign implies that there is a root α in $[0.2, 0.3]$.

ii $f(2.6) = \frac{3}{10} \times 2.6^3 - 2.6^{\frac{2}{3}} + \frac{1}{2.6} - 4$
$= -0.233... < 0$

$f(2.7) = \frac{3}{10} \times 2.7^3 - 2.7^{\frac{2}{3}} + \frac{1}{2.7} - 4$
$= 0.336... > 0$

The change of sign implies that there is a root α in $[2.6, 2.7]$.

b $f(x) = \frac{3}{10}x^3 - x^{\frac{2}{3}} + \frac{1}{x} - 4 = 0$

$$\frac{3}{10}x^3 = 4 + x^{\frac{2}{3}} - \frac{1}{x}$$
$$x^3 = \frac{10}{3}\left(4 + x^{\frac{2}{3}} - \frac{1}{x}\right)$$
$$x = \sqrt[3]{\frac{10}{3}\left(4 + x^{\frac{2}{3}} - \frac{1}{x}\right)}$$

c Using $x_0 = 2.5$:

$$x_1 = \sqrt[3]{\frac{10}{3}\left(4 + 2.5^{\frac{2}{3}} - \frac{1}{2.5}\right)} = 2.6275$$
$$x_2 = \sqrt[3]{\frac{10}{3}\left(4 + 2.6275^{\frac{2}{3}} - \frac{1}{2.6275}\right)} = 2.6406$$
$$x_3 = \sqrt[3]{\frac{10}{3}\left(4 + 2.6406^{\frac{2}{3}} - \frac{1}{2.6406}\right)} = 2.6419$$
$$x_3 = \sqrt[3]{\frac{10}{3}\left(4 + 2.6419^{\frac{2}{3}} - \frac{1}{2.6419}\right)} = 2.6420$$

All correct to 4 decimal places.

d $f'(x) = \frac{9}{10}x^2 - \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{x^2}$

Using $x_0 = 0.3$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0.3 - \frac{f(0.3)}{f'(0.3)}$$
$$= 0.3 - \frac{-1.10671}{-12.02598}$$
$$= 0.208 \text{ (3 d.p.)}$$

20 a $v(x) = 0.12\cos\left(\frac{2x}{5}\right) - 0.35\sin\left(\frac{2x}{5}\right) + 120$

$$= R\left(\cos\frac{2x}{5} + \alpha\right) + 120$$

$$R\left(\cos\frac{2x}{5} + \alpha\right)$$
$$= R\left(\cos\frac{2x}{5}\cos\alpha - \sin\frac{2x}{5}\sin\alpha\right)$$
$$= 0.12\cos\left(\frac{2x}{5}\right) - 0.35\sin\left(\frac{2x}{5}\right)$$

So $R\cos\alpha = 0.12$ and $R\sin\alpha = 0.35$

$$\tan\alpha = \frac{0.35}{0.12} \Rightarrow \alpha = 1.2405 \text{ (4 d.p.)}$$

$$R^2(\cos^2\theta + \sin^2\theta) = 0.12^2 + 0.35^2$$
$$R^2 = 0.1369 \text{ so } R = 0.37$$

20 b $\mathrm{v}(x)=0.37\cos\left(\frac{2x}{5}+1.2405\right)+120$

$\mathrm{v}'(x)=-\frac{2}{5}\times 0.37\sin\left(\frac{2x}{5}+1.2405\right)$

$=-0.148\sin\left(\frac{2x}{5}+1.2405\right)$

c $\mathrm{v}'(4.7)=-0.148\sin\left(\frac{9.4}{5}+1.2405\right)$

$=-0.0031\ldots<0$

$\mathrm{v}'(4.8)=-0.148\sin\left(\frac{9.6}{5}+1.2405\right)$

$=0.0028\ldots>0$

The change of sign implies that there is a stationary point in the interval $[4.7, 4.8]$.

d $\mathrm{v}''(x)=-\frac{2}{5}\times 0.148\cos\left(\frac{2x}{5}+1.2405\right)$

Using $x_0=12.6$:

$x_1=12.6-\frac{\mathrm{v}'(12.6)}{\mathrm{v}''(12.6)}$

$=12.6-\frac{0.148\sin\left(\frac{25.2}{5}+1.2405\right)}{\frac{2}{5}\times 0.148\cos\left(\frac{25.2}{5}+1.2405\right)}$

$=12.6+\frac{0.0003974\ldots}{0.5920\ldots}$

$=12.607$ (3 d.p.)

e $\mathrm{v}'(12.60665)$

$=-0.148\sin\left(\frac{25.2133}{5}+1.2405\right)$

$=0.0000037\ldots>0$

$\mathrm{v}'(12.60675)$

$=-0.148\sin\left(\frac{25.2135}{5}+1.2405\right)$

$=-0.0000022\ldots<0$

The change of sign implies that there is a stationary point at $x=12.6067$ correct to 4 decimal places.

21 $\int_a^3 (12-3x)^2\,\mathrm{d}x=78$

$\left[-\frac{1}{9}(12-3x)^3\right]_a^3=-\frac{27}{9}+\frac{1}{9}(12-3a)^3$

$-3+\frac{1}{9}(12-3a)^3=78$

$\frac{1}{9}(12-3a)^3=81$

$(12-3a)^3=729$

$12-3a=9$

$a=1$

22 a $\cos(5x+2x)=\cos 5x\cos 2x-\sin 5x\sin 2x$

$\cos(5x-2x)=\cos 5x\cos 2x+\sin 5x\sin 2x$

Adding:

$\cos 7x+\cos 3x=2\cos 5x\cos 2x$

b $\int 6\cos 5x\cos 2x\,\mathrm{d}x$

$=3\int(\cos 7x+\cos 3x)\mathrm{d}x$

$=\frac{3}{7}\sin 7x+\sin 3x+c$

23 Consider $y=\mathrm{e}^{x^4}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=4x^3\mathrm{e}^{x^4}$

So $\int_0^m mx^3\mathrm{e}^{x^4}\,\mathrm{d}x=\left[\frac{m}{4}\mathrm{e}^{x^4}\right]_0^m$

$=\frac{m}{4}\mathrm{e}^{m^4}-\frac{m}{4}$

So $\frac{m}{4}\mathrm{e}^{m^4}-\frac{m}{4}=\frac{3}{4}\left(\mathrm{e}^{81}-1\right)$

$\frac{m}{4}\left(\mathrm{e}^{m^4}-1\right)=\frac{3}{4}\left(\mathrm{e}^{81}-1\right)$

$m=3$

24 Let $I = \int_1^5 \frac{3x}{\sqrt{2x-1}}\,dx$

Let $u^2 = 2x - 1 \Rightarrow 2u\frac{du}{dx} = 2$

So replace dx with u du.

$\sqrt{2x-1} = u$ and $x = \frac{u^2+1}{2}$

x	u
1	1
5	3

So $I = \int_1^3 \frac{3}{2} \times \frac{u^2+1}{u} \times u\,du$

$= \int_1^3 \left(\frac{3}{2}u^2 + \frac{3}{2}\right)du$

$= \left[\frac{1}{2}u^3 + \frac{3}{2}u\right]_1^3$

$= \left(\frac{27}{2} + \frac{9}{2}\right) - \left(\frac{1}{2} + \frac{3}{2}\right)$

$= 18 - 2$

$= 16$

25 Let $I = \int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,dx$

Let $u = 1 - x^2 \Rightarrow \frac{du}{dx} = -2x$

So replace x dx with $-\frac{du}{2}$.

$x^2 = 1 - u$

So $\int \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,dx = \int \frac{x^2}{(1-x^2)^{\frac{1}{2}}}\,x\,dx$

$= \int \frac{1-u}{u^{\frac{1}{2}}}\left(-\frac{du}{2}\right)$

$= -\frac{1}{2}\int \frac{1-u}{u^{\frac{1}{2}}}\,du$

$= -\frac{1}{2}\int \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du$

x	u
$\frac{1}{2}$	$\frac{3}{4}$
0	1

So $I = \left[-u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}}\right]_1^{\frac{3}{4}}$

$= \left(-\frac{\sqrt{3}}{2} + \frac{1}{3} \times \frac{3\sqrt{3}}{4\sqrt{4}}\right) - \left(-1 + \frac{1}{3}\right)$

$= \left(-\frac{3\sqrt{3}}{8}\right) - \left(-\frac{2}{3}\right)$

$= \frac{2}{3} - \frac{3\sqrt{3}}{8}$

26 Let $I = \int_1^e (x^2+1)\ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

and $\frac{dv}{dx} = x^2 + 1 \Rightarrow v = \frac{x^3}{3} + x$

Using the integration by parts formula:

$$I = \left[\left(\frac{x^3}{3} + x\right)\ln x\right]_1^e - \int_1^e \frac{1}{x}\left(\frac{x^3}{3} + x\right)dx$$

$$= \left(\frac{e^3}{3} + e\right)\times 1 - \left(\frac{1}{3} + 1\right)\times 0 - \int_1^e \left(\frac{x^2}{3} + 1\right)dx$$

$$= \frac{e^3}{3} + e - 0 - \left[\frac{x^3}{9} + x\right]_1^e$$

$$= \frac{e^3}{3} + e - \left(\left(\frac{e^3}{9} + e\right) - \left(\frac{1}{9} + 1\right)\right)$$

$$= \frac{2e^3}{9} + \frac{10}{9}$$

$$= \frac{1}{9}(2e^3 + 10)$$

27 a $\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{2x-3} + \frac{B}{x+2}$

$$\equiv \frac{A(x+2) + B(2x-3)}{(2x-3)(x+2)}$$

$$5x+3 \equiv A(x+2) + B(2x-3)$$

Let $x = -2$: $\quad -7 = B(-7)$ so $B = 1$

Let $x = \frac{3}{2}$: $\quad \frac{21}{2} = A\left(\frac{7}{2}\right)$ so $A = 3$

So $\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{3}{2x-3} + \frac{1}{x+2}$

b $\int_2^6 \frac{5x+3}{(2x-3)(x+2)} dx$

$$= \int_2^6 \frac{3}{2x-3} dx + \int_2^6 \frac{1}{x+2} dx$$

$$= \left[\frac{3}{2}\ln(2x-3) + \ln(x+2)\right]_2^6$$

$$= \left(\frac{3}{2}\ln 9 + \ln 8\right) - \left(\frac{3}{2}\ln 1 + \ln 4\right)$$

$$= \ln 9^{\frac{3}{2}} + \ln 8 - 0 - \ln 4$$

$$= \ln 9^{\frac{3}{2}} + \ln \frac{8}{4}$$

$$= \ln 27 + \ln 2$$

$$= \ln 54$$

28 a $2\cos t = 1$

$$\Rightarrow \cos t = \frac{1}{2}$$

$$\Rightarrow t = \frac{\pi}{3} \quad \text{or} \quad t = \frac{5\pi}{3}$$

b $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt} dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)(1 - 2\cos t) dt$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 dt$$

c $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 dt$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 4\cos t + 4\cos^2 t \, dt$$

Using double angle formula:

$$\int \cos^2 t \, dt = \int \frac{\cos 2t + 1}{2} dt$$

Using substitution $u = 2t$, $dt = \frac{1}{2} du$:

$$\int \cos 2t \, dt = \frac{\sin 2t}{2}$$

$$\Rightarrow \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 4\cos t + 4\cos^2 t \, dt$$

$$= \left[-4\sin t + \sin 2t + 3t\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$= \left(\frac{3\sqrt{2}}{2} + 5\pi\right) - \left(-\frac{3\sqrt{2}}{2} + \pi\right)$$

$$= 4\pi + 3\sqrt{3}$$

29 a At point A, x coordinate is zero and y is maximum, therefore $t=\frac{\pi}{6}$

Gradient at point A:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt}=a\cos t$$

$$\frac{dx}{dt}=-3a\sin 3t$$

$$\frac{dy}{dx}=\frac{a\cos t}{-3a\sin 3t}=\frac{\cos t}{-3\sin 3t}$$

$$t=\frac{\pi}{6}$$

$$\frac{dy}{dx}=\frac{\sqrt{3}/2}{-3}=-\frac{\sqrt{3}}{6}$$

Equation of AD:

$$y-\frac{1}{2}a=-\frac{\sqrt{3}}{6}(x-0)$$

$$y=-\frac{\sqrt{3}}{6}x+\frac{1}{2}a$$

$x-$coordinate of D:

$$0=-\frac{\sqrt{3}}{6}x+\frac{1}{2}a$$

$$x=\frac{1}{2}a\left(\frac{6}{\sqrt{3}}\right)=a\sqrt{3}$$

Area:

$$=\int_0^{a\sqrt{3}}-\frac{\sqrt{3}}{6}x+\frac{1}{2}a\ dx-\int_0^{\frac{\pi}{6}}a\sin t(-3a\sin 3t)\,dt$$

$$=\left[-\frac{\sqrt{3}}{12}x^2+\frac{1}{2}ax\right]_0^{a\sqrt{3}}-3a^2\int_0^{\frac{\pi}{6}}\sin t\sin 3t\ dt$$

Using product to sum formulas:

$$=\left[-\frac{\sqrt{3}}{12}x^2+\frac{1}{2}ax\right]_0^{a\sqrt{3}}-3a^2\int_0^{\frac{\pi}{6}}-\frac{\cos 4t-\cos 2t}{2}\ dt$$

$$=\left[-\frac{\sqrt{3}}{12}x^2+\frac{1}{2}ax\right]_0^{a\sqrt{3}}-3a^2\left[\frac{\sin 2t}{4}-\frac{\sin 4t}{8}\right]_0^{\frac{\pi}{6}}$$

$$=\left(\left(\frac{\sqrt{3}}{4}a^2\right)-0\right)-3a^2\left(\left(\frac{\sqrt{3}}{16}\right)-0\right)$$

$$=\frac{\sqrt{3}}{4}a^2-\frac{3\sqrt{3}}{16}a^2$$

$$=\frac{\sqrt{3}}{16}a^2$$

b $$2\left(\frac{\sqrt{3}}{16}a^2\right)=10$$

$$a^2=\frac{80}{\sqrt{3}}$$

$$a=\sqrt{\frac{80}{\sqrt{3}}}$$

$$=6.796\text{ (4 s.f.)}$$

30 a Let $I = \int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x$

Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \frac{1}{2}\mathrm{e}^{2x}$

Using the integration by parts formula:

$$I = \left[\frac{1}{2}x\mathrm{e}^{2x}\right]_0^1 - \int_0^1 1\times\frac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x$$

$$= \left(\frac{1}{2}\mathrm{e}^2 - 0\right) - \left[\frac{1}{4}\mathrm{e}^{2x}\right]_0^1$$

$$= \frac{1}{2}\mathrm{e}^2 - \left(\frac{1}{4}\mathrm{e}^2 - \frac{1}{4}\right)$$

$$= \frac{1}{4}\mathrm{e}^2 + \frac{1}{4}$$

b When $x = 0.4$, $y = 0.89022$
When $x = 0.8$, $y = 3.96243$

c Area of R

$$\approx \tfrac{1}{2}\times 0.2(0 + 2(0.29836 + 0.89022 + 1.99207 + 3.96243) + 7.38906)$$

$$= 0.1 \times 21.67522$$

$$= 2.168\ (4\text{s.f.})$$

d Percentage error in answer from part **c**

$$= \frac{\frac{1}{4}\mathrm{e}^2 + \frac{1}{4} - 2.168}{\frac{1}{4}\mathrm{e}^2 + \frac{1}{4}} \times 100\% = 3.37\%$$

31 a $\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{A}{x-1} + \frac{B}{2x-3}$

$$2x-1 \equiv A(2x-3) + B(x-1)$$

Let $x = \frac{3}{2}$: $\quad 2 = B\left(\frac{1}{2}\right) \Rightarrow B = 4$

Let $x = 1$: $\quad 1 = A(-1) \Rightarrow A = -1$

So $\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{-1}{x-1} + \frac{4}{2x-3}$

b $(2x-3)(x-1)\frac{\mathrm{d}y}{\mathrm{d}x} = (2x-1)y$

Separating the variables:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{2x-1}{(2x-3)(x-1)}\,\mathrm{d}x$$

So $\ln y = \int \frac{-1}{x-1}\,\mathrm{d}x + \int \frac{4}{2x-3}\,\mathrm{d}x$

$$= -\ln|x-1| + 2\ln|2x-3| + c$$

$$= -\ln|x-1| + \ln(2x-3)^2 + \ln A$$

$$= \ln A\frac{(2x-3)^2}{x-1}$$

So the general solution is

$$y = \frac{A(2x-3)^2}{x-1}$$

c $y = \frac{A(2x-3)^2}{x-1}$

When $x = 2$, $y = 10$ so

$$10 = \frac{A(4-3)^2}{2-1} \Rightarrow A = 10$$

So the particular solution is

$$y = \frac{10(2x-3)^2}{(x-1)}$$

32 a $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

Using the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$
$$= 4\pi r^2 \times \frac{dr}{dt}$$

So $\frac{k}{V} = 4\pi r^2 \times \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{k}{\frac{4}{3}\pi r^3} \times \frac{1}{4\pi r^2}$$
$$= \frac{3k}{16\pi^2 r^5}$$

So $B = \frac{3k}{16\pi^2}$

b Separating the variables:

$$\int r^5 dr = \int \frac{3k}{16\pi^2} dt$$
$$\frac{r^6}{6} = \frac{3k}{16\pi^2} t + A$$
$$r^6 = \frac{9k}{8\pi^2} t + A'$$
$$r = \left(\frac{9k}{8\pi^2} t + A' \right)^{\frac{1}{6}}$$

33 a Rate of change of volume is $\frac{dV}{dt}$ cm^3 s^{-1}

Increase is 20 cm^3 s^{-1}
Decrease is kV cm^3 s^{-1}, where k is a constant of proportionality.

So the overall rate of change is

$$\frac{dV}{dt} = 20 - kV$$

b Separating the variables:

$$\int \frac{1}{20 - kV} dV = \int 1 dt$$

So $-\frac{1}{k} \ln |20 - kV| = t + c$

When $t = 0$, $V = 0$ so

$$-\frac{1}{k} \ln 20 = c$$

Combining the ln terms:

$$-\frac{1}{k} \ln \frac{20 - kV}{20} = t$$
$$\ln \frac{20 - kV}{20} = -kt$$
$$\frac{20 - kV}{20} = e^{-kt}$$
$$kV = 20 - 20e^{-kt}$$
$$V = \frac{20}{k} - \frac{20}{k} e^{-kt}$$

So $A = \frac{20}{k}$ and $B = -\frac{20}{k}$

33 c $V = \frac{20}{k} - \frac{20}{k}e^{-kt} \Rightarrow \frac{dV}{dt} = 20e^{-kt}$

Substitute $\frac{dV}{dt} = 10$ when $t = 5$:

$10 = 20e^{-5k} \Rightarrow e^{-5k} = \frac{1}{2}$

Taking natural logarithms:

$-5k = \ln\frac{1}{2}$ or $5k = \ln 2$

$k = \frac{1}{5}\ln 2 = 0.1386$ (4 d.p.)

So $V = \frac{100}{\ln 2} - \frac{100}{\ln 2}\left(\frac{1}{2}\right)^{\frac{t}{5}}$

When $t = 10$:

$V = \frac{100}{\ln 2} - \frac{100}{\ln 2} \times \frac{1}{4}$

$= \frac{75}{\ln 2}$

$= 108.2$ (1 d.p.)

So the volume is 108 cm^3 (3 s.f.).

34 a $\frac{dC}{dt}$ is the rate of change of concentration.

The concentration is decreasing so the rate of change is negative.

So $-\frac{dC}{dt} \propto C$ or $\frac{dC}{dt} = -kC$,
where k is a positive constant of proportionality.

b Separating the variables:

$\int \frac{1}{C}\,dC = -\int k\,dt$

so $\ln C = -kt + \ln A$,
where $\ln A$ is a constant.

So $\ln\frac{C}{A} = -kt$

$\frac{C}{A} = e^{-kt}$

So the general solution is $C = Ae^{-kt}$.

c When $t = 0, C = C_0$ so $A = C_0$

So $C = C_0 e^{-kt}$

When $t = 4, C = \frac{1}{10}C_0$ so

$\frac{1}{10}C_0 = C_0 e^{-4k}$

$e^{4k} = 10$

$4k = \ln 10$

$k = \frac{1}{4}\ln 10$

35 $|\overrightarrow{PQ}| = \sqrt{(8-(-1))^2 + (-4-4)^2 + (k-6)^2}$

$= \sqrt{9^2 + (-8)^2 + (k-6)^2} = 7\sqrt{5}$

$81 + 64 + (k-6)^2 = 245$

$(k-6)^2 = 100$

$k - 6 = \pm 10$

$k = -4$ or $k = 16$

36 $|\overrightarrow{AB}| = \sqrt{1+36+16} = \sqrt{53}$

$|\overrightarrow{AC}| = \sqrt{25+4+9} = \sqrt{38}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 6\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$

$|\overrightarrow{BC}| = \sqrt{36+64+49} = \sqrt{149}$

$\cos\angle BAC = \frac{53+38-149}{2\times\sqrt{53}\times\sqrt{38}} = -0.6462...$

$\angle BAC = 130.3°$ (1 d.p.)

37 a Let O be the fixed origin.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 10\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{100+25+4} = \sqrt{129}$

Unit vector in direction of $\overrightarrow{PQ}$

$= \frac{10}{\sqrt{129}}\mathbf{i} - \frac{5}{\sqrt{129}}\mathbf{j} - \frac{2}{\sqrt{129}}\mathbf{k}$

c $\cos\theta_z = \frac{-2}{\sqrt{129}} = -0.1761$

$\theta_z = 101.1°$ (1 d.p.)

37 d $\overrightarrow{AB} = 30\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$

There is no scalar, say m, for which $\overrightarrow{AB} = m\overrightarrow{PQ}$, so $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are not parallel.

38 $\overrightarrow{MN} = 10\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$

$\left|\overrightarrow{MN}\right| = \sqrt{10^2 + 5^2 + 4^2} = \sqrt{141}$

$\overrightarrow{MP} = (k+2)\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}$

$\left|\overrightarrow{MP}\right| = \sqrt{(k+2)^2 + 2^2 + 11^2}$

$= \sqrt{(k+2)^2 + 125}$

$\overrightarrow{NP} = (k-8)\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$

$\left|\overrightarrow{NP}\right| = \sqrt{(k-8)^2 + 3^2 + 7^2}$

$= \sqrt{(k-8)^2 + 56}$

If $\left|\overrightarrow{MN}\right| = \left|\overrightarrow{MP}\right|$ then

$\sqrt{141} = \sqrt{(k+2)^2 + 125}$

$(k+2)^2 = 16$

$k + 2 = \pm 4$

$k = 2$ or $k = -6$

$\Rightarrow k = 2$ (since k is positive)

If $\left|\overrightarrow{MN}\right| = \left|\overrightarrow{NP}\right|$ then

$\sqrt{141} = \sqrt{(k-8)^2 + 56}$

$(k-8)^2 = 85$

So there are no integer solutions for k if $\left|\overrightarrow{MN}\right| = \left|\overrightarrow{NP}\right|$

If $\left|\overrightarrow{MP}\right| = \left|\overrightarrow{NP}\right|$ then

$\sqrt{(k+2)^2 + 125} = \sqrt{(k-8)^2 + 56}$

$k^2 + 4k + 129 = k^2 - 16k + 122$

$20k = -7$

So there are no positive solutions for k if $\left|\overrightarrow{MP}\right| = \left|\overrightarrow{NP}\right|$

So $k = 2$

39 $-6\mathbf{i} + 40\mathbf{j} + 16\mathbf{k} = 3p\mathbf{i} + (8 + qr)\mathbf{j} + 2pr\mathbf{k}$

Comparing coefficients of **i**:
$-6 = 3p \Rightarrow p = -2$

Comparing coefficients of **k**:
$16 = 2pr \Rightarrow pr = 8 \Rightarrow r = -4$

Comparing coefficients of **j**:
$40 = 8 + qr \Rightarrow qr = 32 \Rightarrow q = -8$

$p = -2, q = -8, r = -4$

Challenge

1 a $ay + x^2 + 4xy = y^2$

Differentiating with respect to x:

$$a\frac{dy}{dx} + 2x + 4\left(x\frac{dy}{dx} + y\right) = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}(a + 4x - 2y) = -4y - 2x$$

$$= \frac{-4y - 2x}{a + 4x - 2y}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = -2y$$

Substituting for x in the original equation:

$$ay + 4y^2 - 8y^2 = y^2$$

$$ay - 5y^2 = 0$$

$$y(a - 5y) = 0 \Rightarrow y = 0 \text{ or } y = \frac{a}{5}$$

When $y = 0$, $x = -2y = 0$

When $y = \frac{a}{5}$, $x = -2y = \frac{-2a}{5}$

So $\frac{dy}{dx} = 0$ at $(0, 0)$ and at $\left(-\frac{2a}{5}, \frac{a}{5}\right)$.

b $$\frac{dx}{dy} = \frac{a + 4x - 2y}{-4y - 2x}$$

$$\frac{dx}{dy} = 0 \Rightarrow a + 4x - 2y = 0 \Rightarrow y = 2x + \frac{a}{2}$$

Substituting for y in the original equation:

$$a\left(2x + \frac{a}{2}\right) + x^2 + 4x\left(2x + \frac{a}{2}\right) = \left(2x + \frac{a}{2}\right)^2$$

$$2ax + \frac{a^2}{2} + x^2 + 8x^2 + 2ax = 4x^2 + 2ax + \frac{a^2}{4}$$

$$5x^2 + 2ax + \frac{a^2}{4} = 0$$

$$\text{‘}b^2 - 4ac\text{’} = 4a^2 - \frac{20a^2}{4} = -a^2$$

$-a^2 < 0$ (as $a \neq 0$) so $5x^2 + 2ax + \frac{a^2}{4} = 0$ has no solutions.

Hence $\frac{dx}{dy} \neq 0$ for all x.

2 $y = \sin x + 2$ and $y = \cos 2x + 2$

Curves intersect when
$\sin x + 2 = \cos 2x + 2$

$\sin x = \cos 2x$
$= 1 - 2\sin^2 x$
$2\sin^2 x + \sin x - 1 = 0$
$(2\sin x - 1)(\sin x + 1) = 0$

So $\sin x = \frac{1}{2}$ or $\sin x = -1$

So the intersections are at
$x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ and $x = \frac{3\pi}{2}$

Shaded area up to $x = \frac{\pi}{6}$ is

$$\int_0^{\frac{\pi}{6}} \left(\cos 2x + 2 - (\sin x + 2)\right) dx$$
$$= \int_0^{\frac{\pi}{6}} (\cos 2x - \sin x) dx$$
$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_0^{\frac{\pi}{6}}$$
$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0 + 1)$$
$$= \frac{3\sqrt{3}}{4} - 1$$

Shaded area between $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ is

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\sin x - \cos 2x) dx$$
$$= \left[-\cos x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$
$$= \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right)$$
$$= \frac{3\sqrt{3}}{2}$$

Shaded area between $x = \frac{5\pi}{6}$ and $\frac{3\pi}{2}$ is

$$\int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (\cos 2x - \sin x) dx$$
$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{\frac{5\pi}{6}}^{\frac{3\pi}{2}}$$
$$= (0 + 0) - \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{3\sqrt{3}}{4}$$

So the total shaded area is

$$\frac{3\sqrt{3}}{4} - 1 + \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}$$
$$= 3\sqrt{3} - 1$$