Variable acceleration 11D

1 a $s = \int v\,dt$

$= \int (3t^2 - 1)\,dt$

$= t^3 - t + c$, where c is a constant of integration.

When $t = 0$, $x = 0$:
$0 = 0 - 0 + c \Rightarrow c = 0$
$s = t^3 - t$

b $s = \int v\,dt$

$= \int \left(2t^3 - \frac{3t^2}{2}\right)dt$

$= \frac{t^4}{2} - \frac{t^3}{2} + c$, where c is a constant of integration.

When $t = 0$, $x = 0$:
$0 = 0 - 0 + c \Rightarrow c = 0$
$s = \frac{t^4}{2} - \frac{t^3}{2}$

c $s = \int v\,dt$

$= \int (2\sqrt{t} + 4t^2)\,dt$

$= \frac{4}{3}t^{\frac{3}{2}} + \frac{4t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $x = 0$:
$0 = 0 + 0 + c \Rightarrow c = 0$
$s = \frac{4}{3}t^{\frac{3}{2}} + \frac{4t^3}{3}$

2 a $v = \int a\,dt$

$= \int (8t - 2t^2)\,dt$

$= 4t^2 - \frac{2t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $v = 0$:
$0 = 0 - 0 + c \Rightarrow c = 0$
$v = 4t^2 - \frac{2t^3}{3}$

b $v = \int a\,dt$

$= \int \left(6 + \frac{t^2}{3}\right)dt$

2 b $v = 6t + \frac{t^3}{9} + c$, where c is a constant of integration.

When $t = 0$, $v = 0$:
$0 = 0 + 0 + c \Rightarrow c = 0$
$v = 6t + \frac{t^3}{9}$

3 $x = \int v \, dt$
$= \int (8 + 2t - 3t^2) \, dt$
$= 8t + t^2 - t^3 + c$, where c is a constant of integration.

When $t = 0$, $x = 4$:
$4 = 0 + 0 - 0 + c \Rightarrow c = 4$
$x = 8t + t^2 - t^3 + 4$

When $t = 1$,
$x = 8 + 1 - 1 + 4 = 12$

The distance of P from O when $t = 1$ is 12 m.

4 a $v = \int a \, dt$
$= \int (16 - 2t) \, dt$
$= 16t - t^2 + c$, where c is a constant of integration.

When $t = 0$, $v = 6$:
$6 = 0 - 0 + c \Rightarrow c = 6$
$v = 16t - t^2 + 6$

b $x = \int v \, dt$
$= \int (16t - t^2 + 6) \, dt$
$= 8t^2 - \frac{t^3}{3} + 6t + k$, where k is a constant of integration.

When $t = 3$, $x = 75$:
$75 = 8 \times 3^2 - \frac{3^3}{3} + 6 \times 3 + k$
$\Rightarrow k = 75 - 72 + 9 - 18 = -6$
$x = 8t^2 - \frac{t^3}{3} + 6t - 6$

4 b When $t = 0$,
$x = 0 - 0 + 0 - 6 = -6$

5 $v = 6t^2 - 51t + 90$
P is at rest when $v = 0$.
$6t^2 - 51t + 90 = 0$

$2t^2 - 17t + 30 = 0$

$(2t-5)(t-6)=0$

P is at rest when $t = 2.5$ and when $t = 6$.

$$s=\int_{2.5}^{6}(6t^2-51t+90)\,\mathrm{d}t$$
$$=\left[2t^3-\frac{51t^2}{2}+90t\right]_{2.5}^{6}$$
$$=\left(2\times6^3-\frac{51\times6^2}{2}+90\times6\right)-\left(2\times2.5^3-\frac{51\times2.5^2}{2}+90\times2.5\right)$$
$$=(432-918+540)-(31.25-159.375+225)$$
$$=-42.875\ldots$$
$$=-42.9\text{ (3 s.f.)}$$

The negative sign indicates that the displacement is negative, but this can be ignored as distance is required.

The distance between the two points where P is at rest is 42.9 m (3 s.f.).

6 $s=\int v\,\mathrm{d}t$

$$=\int(12+t-6t^2)\,\mathrm{d}t$$
$$=12t+\frac{t^2}{2}-2t^3+c$$, where c is a constant of integration.

When $t = 0$, $s = 0$:
$0=0+0-0+c\Rightarrow c=0$
$$s=12t+\frac{t^2}{2}-2t^3$$

$v = 0$ when
$12+t-6t^2=0$
$(3-2t)(4+3t)=0$
$t>0$, so $t=1.5$

When $t = 1.5$, $s=12\times1.5+\frac{1.5^2}{2}-2\times1.5^3$
$$=12.375\ldots$$
$$=12.4\text{ (3 s.f.)}$$

The distance of P from O when $v = 0$ is 12.4 m.

7 a $v = 4t - t^2$
P is at rest when $v = 0$.
$4t - t^2 = 0$
$t(4 - t) = 0$
$t > 0$, so $t = 4$

$x = \int v\,dt$
$= \int (4t - t^2)\,dt$
$= 2t^2 - \frac{t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $x = 0$
$0 = 0 + 0 + c \Rightarrow c = 0$
$x = 2t^2 - \frac{t^3}{3}$

When $t = 4$, $x = 2 \times 4^2 - \frac{4^3}{3}$
$= 10\frac{2}{3}$

b When $t = 5$, $x = 2 \times 5^2 - \frac{5^3}{3}$
$= 8\frac{1}{3}$

In the interval $0 \le t \le 5$, P moves to a point $10\frac{2}{3}$ m from O and then returns to a point $8\frac{1}{3}$ m from O.

The total distance moved is $10\frac{2}{3} + (10\frac{2}{3} - 8\frac{1}{3}) = 13$ m.

8 $x = \int v\,dt$
$= \int (6t^2 - 26t + 15)\,dt$
$= 2t^3 - 13t^2 + 15t + c$, where c is a constant of integration.

When $t = 0$, $x = 0$
$0 = 0 - 0 + 0 + c \Rightarrow c = 0$
$x = 2t^3 - 13t^2 + 15t$
$= t(2t^2 - 13t + 15)$
$= t(2t - 3)(t - 5)$

When $x = 0$ and t is non-zero, $t = 1.5$ or $t = 5$

P is again at O when $t = 1.5$ and $t = 5$.

9 a $x = \int v\,dt$
$= \int (3t^2 - 12t + 5)\,dt$
$= t^3 - 6t^2 + 5t + c$, where c is a constant of integration.

9 a When $t = 0$, $x = 0$

$0 = 0 - 0 + 0 + c \Rightarrow c = 0$

$x = t^3 - 6t^2 + 5t$

P returns to O when $x = 0$.

$t^3 - 6t^2 + 5t = 0$

$t(t^2 - 6t + 5) = 0$

$t(t - 1)(t - 5) = 0$

P returns to O when $t = 1$ and $t = 5$.

b $v = 0$ when

$3t^2 - 12t + 5 = 0$

$$t = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(5)}}{6}$$

$= 0.473, 3.52$

So P does not turn round in the interval $2 \leq t \leq 3$.

When $t = 2$,

$x = 2^3 - 6 \times 2^2 + 5 \times 2$

$= 8 - 24 + 10$

$= -6$

When $t = 3$,

$x = 3^3 - 6 \times 3^2 + 5 \times 3$

$= 27 - 54 + 15$

$= -12$

The distance travelled by P in the interval $2 \leq t \leq 3$ is 6 m.

10 $v = \int a \, dt$

$= \int (4t - 3) \, dt$

$= 2t^2 - 3t + c$, where c is a constant of integration.

When $t = 0$, $v = 4$

$4 = 0 - 0 + c \Rightarrow c = 4$

$v = 2t^2 - 3t + 4$,

When $t = T$, $v = 4$ again

$4 = 2T^2 - 3T + 4$

$2T^2 - 3T = 0$

$T(2T - 3) = 0$

$T \neq 0$, so $T = 1.5$

11 a $v = \int a \, dt$

$= \int (t - 3) \, dt$

$= \frac{t^2}{2} - 3t + c$,

11 a When $t = 0$, $v = 4$

$4 = 0 - 0 + c \Rightarrow c = 4$

$$v = \frac{t^2}{2} - 3t + 4$$

b P is at rest when $v = 0$.

$$\frac{t^2}{2} - 3t + 4 = 0$$

$t^2 - 6t + 8 = 0$

$(t - 2)(t - 4) = 0$

$t = 2$ or $t = 4$

P is at rest when $t = 2$ and $t = 4$.

c
$$s = \int_2^4 \left(\frac{t^2}{2} - 3t + 4\right) dt$$
$$= \left[\frac{t^3}{6} - \frac{3t^2}{2} + 4t\right]_2^4$$
$$= \left(\frac{4^3}{6} - \frac{3\times4^2}{2} + 4\times4\right) - \left(\frac{2^3}{6} - \frac{3\times2^2}{2} + 4\times2\right)$$
$$= \left(\frac{32}{3} - 24 + 16\right) - \left(\frac{4}{3} - 6 + 8\right)$$
$$= \frac{8}{3} - \frac{10}{3}$$
$$= -\tfrac{2}{3}$$

The negative sign indicates that the displacement is negative, but this can be ignored as distance is required. The distance between the two points where P is at rest is $\tfrac{2}{3}$ m.

12
$$v = \int a\,dt$$
$$= \int (6t + 2)\,dt$$
$= 3t^2 + 2t + c$, where c is a constant of integration.

$$s = \int v\,dt$$
$$= \int (3t^2 + 2t + c)\,dt$$
$= t^3 + t^2 + ct + k$, where k is a constant of integration.

When $t = 2$, $s = 10$

$10 = 2^3 + 2^2 + 2c + k$

$2c + k = -2$ **(1)**

When $t = 3$, $s = 38$

$38 = 3^3 + 3^2 + 3c + k$

$3c + k = 2$ **(2)**

(2) − **(1)**:

$c = 4$

12 Substituting $c = 4$ into (**1**):
$2 \times 4 + k = -2$
$k = -10$

So the equations are:
$v = 3t^2 + 2t + 4$
$s = t^3 + t^2 + 4t - 10$

a When $t = 4$
$s = 4^3 + 4^2 + 4 \times 4 - 10$
$= 64 + 16 + 16 - 10$
$= 86$

When $t = 4$ s the displacement is 86 m.

b When $t = 4$
$v = 3 \times 4^2 + 2 \times 4 + 4$
$= 48 + 8 + 4$
$= 60$

When $t = 4$ s the velocity is 60 m s^{-1}.

Challenge

At $t = k$, the velocity given by both equations is identical, so:

$$\frac{k^2}{2} + 2 = 10 + \frac{k}{3} - \frac{k^2}{12}$$

$$6k^2 + 24 = 120 + 4k - k^2$$

$$7k^2 - 4k - 96 = 0$$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{4^2 + 4 \times 7 \times 96}}{2 \times 7}$$

$$= \frac{4 \pm 52}{14}$$

$k > 0$, so $k = 4$

For first part of the motion, up to $t = 4$, $s = s_1$

$$s_1 = \int_0^4 \left(\frac{t^2}{2} + 2 \right) dt$$

$$= \left[\frac{t^3}{6} + 2t \right]_0^4$$

$$= \left(\frac{4^3}{6} + 2 \times 4 \right) - \left(\frac{0^3}{6} + 0 \times 4 \right)$$

$$= \frac{56}{3}$$

For second part of the motion, from $t = 4$ to $t = 10$, $s = s_2$

$$s_2 = \int_4^{10} \left(10 + \frac{t}{3} - \frac{t^2}{12} \right) dt$$

$$= \left[10t + \frac{t^2}{6} - \frac{t^3}{36} \right]_4^{10}$$

$$= \left(10 \times 10 + \frac{10^2}{6} - \frac{10^3}{36} \right) - \left(10 \times 4 + \frac{4^2}{6} - \frac{4^3}{36} \right)$$

$$= \frac{800}{9} - \frac{368}{9}$$

$$= 48$$

Total distance $= s_1 + s_2$

$$= \frac{56}{3} + 48$$

$$= \frac{200}{3}$$

The total distance travelled by the arm is $\frac{200}{3}$ m.