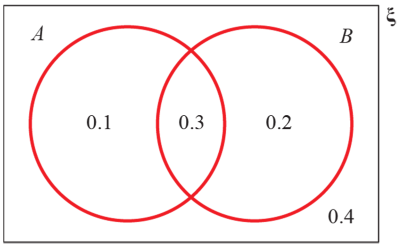
Conditional probability 2D

1 **a** Rewrite the addition formula to obtain

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.5 - 0.6 = 0.3$$

Use this result to complete a Venn diagram to help answer the remaining parts of the question.



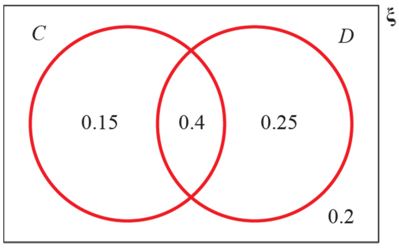
b $P(A') = 0.2 + 0.4 = 0.6$

c $P(A \cup B') = 0.4 + 0.4 = 0.8$

d $P(A' \cup B) = 0.5 + 0.4 = 0.9$

2 **a** $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 0.55 + 0.65 - 0.4 = 0.8$

b



i The required region is the part 'outside' of C and D, which can be found since all of the probabilities must sum to 1.

$$P(C' \cap D') = 1 - P(C \cup D) = 1 - 0.8 = 0.2$$

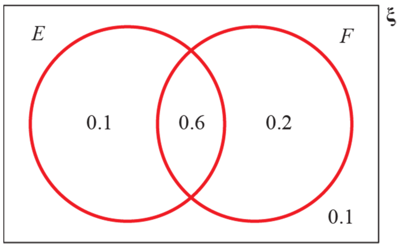
ii $P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{0.4}{0.65} = 0.615$ (3 s.f.)

iii $P(C|D') = \frac{P(C \cap D')}{P(D')} = \frac{0.15}{0.35} = \frac{3}{7} = 0.429$ (3 s.f.)

c From part **bii**, it is known that $P(C|D) \neq P(C)$ so the two events are not independent.
Alternatively, show that $P(C) \times P(D) \neq P(C \cap D)$.

3 a $P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.7 + 0.8 - 0.6 = 0.9$

b



i The required region is within E as well as everything outside F. It includes three of the four regions in the Venn diagram.
$P(E \cup F') = 0.1 + 0.6 + 0.1 = 0.8$

ii The required region is that part of F that does not intersect E.
$P(E \cap F') = 0.2$

iii $P(E|F') = \dfrac{P(E \cap F')}{P(F')} = \dfrac{0.1}{0.1 + 0.1} = \dfrac{1}{2} = 0.5$

4 a
$$\begin{aligned} P(T \cup Q) &= P(T) + P(Q) - P(T \cap Q) \\ 0.75 &= 3P(T \cap Q) + 3P(T \cap Q) - P(T \cap Q) \\ 5P(T \cap Q) &= 0.75 \\ P(T \cap Q) &= 0.15 \end{aligned}$$

b As P(T) = P(Q), using $P(T \cup Q) = P(T) + P(Q) - P(T \cap Q)$ gives

$0.75 = 2P(T) - 0.15$

$\Rightarrow 2P(T) = 0.9$

$\Rightarrow P(T) = 0.45$

c $P(Q') = 1 - P(Q) = 1 - P(T) = 1 - 0.45 = 0.55$

d $P(T' \cap Q') = 1 - P(T \cup Q) = 1 - 0.75 = 0.25$

e $P(T \cap Q') = P(T) - P(T \cap Q) = 0.45 - 0.15 = 0.3$

5 Let F be the event has a freezer and D be the event has a dishwasher. The question requires finding $P(F \cap D)$. Use the addition formula
$P(F \cap D) = P(F) + P(D) - P(F \cup D) = 0.7 + 0.2 - 0.8 = 0.1$

6 a Use the multiplication formula for conditional probability to find $\mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times \mathrm{P}(B) = 0.4 \times 0.5 = 0.2$

Now use the multiplication formula again to find $\mathrm{P}(B|A)$

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{0.2}{0.4} = \frac{1}{2} = 0.5$$

b Use the addition formula to find $\mathrm{P}(A \cup B)$

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.4 + 0.5 - 0.2 = 0.7$

Now $\mathrm{P}(A' \cap B')$ can be found as it is the region outside $\mathrm{P}(A \cup B)$

$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B) = 1 - 0.7 = 0.3$

c $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.5 - 0.2 = 0.3$

7 a First use the addition formula to find $\mathrm{P}(A \cap B)$

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = \frac{1}{4} + \frac{1}{2} - \frac{3}{5} = \frac{3}{20}$$

Now use the multiplication formula to

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\frac{3}{20}}{\frac{1}{2}} = \frac{3}{10} = 0.3$$

b $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{1}{2} - \frac{3}{20} = \frac{7}{20} = 0.35$

c $\mathrm{P}(A' \cap B') = 1 - P(A \cup B) = 1 - \frac{3}{5} = \frac{2}{5} = 0.4$

8 a $\mathrm{P}(C \cap D) = \mathrm{P}(C|\mathrm{D}) \times \mathrm{P}(D) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} = 0.0833$ (3 s.f.)

b $\mathrm{P}(C \cap D') = \mathrm{P}(C|\mathrm{D}') \times \mathrm{P}(D') = \frac{1}{5} \times \left(1 - \frac{1}{4}\right) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20} = 0.15$

c $\mathrm{P}(C) = \mathrm{P}(C \cap D') + \mathrm{P}(C \cap D) = \frac{3}{20} + \frac{1}{12} = \frac{9}{60} + \frac{5}{60} = \frac{14}{60} = \frac{7}{30} = 0.233$ (3 s.f.)

d $\mathrm{P}(D|C) = \frac{\mathrm{P}(D \cap C)}{\mathrm{P}(C)} = \frac{\frac{1}{12}}{\frac{7}{30}} = \frac{30}{84} = \frac{5}{14} = 0.357$ (3 s.f.)

e $\mathrm{P}(D'|C) = 1 - \mathrm{P}(D|C) = 1 - \frac{5}{14} = \frac{9}{14} = 0.643$ (3 s.f.)

8 f $\mathrm{P}(D'|C') = \dfrac{\mathrm{P}(C' \cap D')}{\mathrm{P}(C')} = \dfrac{1-\mathrm{P}(C \cup D)}{\mathrm{P}(C')}$

However, $\mathrm{P}(C') = 1 - \mathrm{P}(C) = \dfrac{7}{30} = \dfrac{23}{30}$

And $\mathrm{P}(C \cup D) = \mathrm{P}(C) + \mathrm{P}(D) - \mathrm{P}(C \cap D) = \dfrac{7}{30} + \dfrac{1}{4} - \dfrac{1}{12} = \dfrac{24}{60} = \dfrac{2}{5}$

So $\mathrm{P}(D'|C') = \dfrac{\mathrm{P}(C' \cap D')}{\mathrm{P}(C')} = \dfrac{1-\mathrm{P}(C \cup D)}{\mathrm{P}(C')} = \dfrac{1-\frac{2}{5}}{\frac{23}{30}} = \dfrac{3}{5} \times \dfrac{30}{23} = \dfrac{18}{23} = 0.783 \text{ (3 s.f.)}$

9 a $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.42 + 0.37 - 0.12 = 0.67$

b $\mathrm{P}(A|B') = \dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \dfrac{\mathrm{P}(A) - \mathrm{P}(A \cap B)}{1 - \mathrm{P}(B)} = \dfrac{0.42 - 0.12}{1 - 0.37} = \dfrac{0.3}{0.63} = 0.476 \text{ (3 s.f.)}$

c Since the events A and C are independent, $\mathrm{P}(A \cap C) = \mathrm{P}(A) \times \mathrm{P}(C) = 0.42 \times 0.3 = 0.126$

d Since B and C are mutually exclusive, there is no need to have an intersection between B and C on the diagram. Work out the probabilities associated with each region as follows:

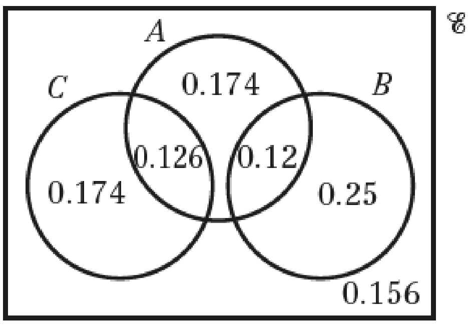
$\mathrm{P}(C \cap A') = \mathrm{P}(C) - \mathrm{P}(A \cap C) = 0.3 - 0.126 = 0.174$

$\mathrm{P}(B \cap A') = \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.37 - 0.12 = 0.25$

$\mathrm{P}(A \cap B' \cap C') = \mathrm{P}(A) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) = 0.42 - 0.12 - 0.126 = 0.174$

$\mathrm{P}(A \cup B \cup C) = 0.174 + 0.126 + 0.174 + 0.12 + 0.25 = 0.844$

$\mathrm{P}(A' \cap B' \cap C') = 1 - \mathrm{P}(A \cup B \cup C) = 1 - 0.844 = 0.156$



e $\mathrm{P}((A' \cup C)') = 1 - \mathrm{P}(A' \cup C)$

Use the Venn diagram to find $\mathrm{P}(A' \cup C) = 0.174 + 0.126 + 0.25 + 0.156 = 0.706$

So $\mathrm{P}((A' \cup C)') = 1 - 0.706 = 0.294$

10 a B and C are independent: $P(B\cap C)=P(B)\times P(C)=0.7\times 0.4=0.28$

b Using part **a**, $P(B|C)=\dfrac{P(B\cap C)}{P(C)}=\dfrac{0.28}{0.4}=\dfrac{7}{10}=0.7$

c $P(A|B')=\dfrac{P(A\cap B')}{P(B')}=\dfrac{P(A)-P(A\cap B)}{1-P(B)}=\dfrac{0.4-0.3}{1-0.7}=\dfrac{0.1}{0.3}=0.333$ (3 s.f.)

d $P((B\cap C)|A')=\dfrac{P((B\cap C)\cap A')}{P(A')}=\dfrac{P(B\cap C)-P(A\cap B\cap C)}{1-P(A)}$

As A and C are mutually exclusive, $P(A\cap B\cap C)=0$

So $P((B\cap C)|A')=\dfrac{P(B\cap C)}{1-P(A)}=\dfrac{0.28}{1-0.4}=\dfrac{0.28}{0.6}=0.467$ (3 s.f.)

11 a This requires finding $P(A\cap B)$
First find $P(A\cup B)$
$P(A\cup B)=0.9$ as $P(A\cup B)+P(A'\cap B')=1$
Using the addition rule gives
$P(A\cap B)=P(A)+P(B)-P(A\cup B)=0.3+0.7-0.9=0.1$

b This requires finding $P(A|B)$
$P(A|B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.1}{0.7}=0.143$ (3 s.f.)

c Test whether the events are independent
$P(A)\times P(B)=0.3\times 0.7=0.21$, $P(A\cap B)=0.1$
So the events are not independent. If Anna is late, Bella is *less* likely to be late and vice versa.

12 a The probability that both John and Kayleigh win their matches is $P(J\cap K)$
$P(J\cap K)=P(J)+P(K)-P(J\cup K)=0.6+0.7-0.8=0.5$

b $P(J|K')=\dfrac{P(J\cap K')}{P(K')}=\dfrac{P(J)-P(J\cap K)}{1-P(K)}=\dfrac{0.6-0.5}{1-0.7}=\dfrac{0.1}{0.3}=0.333$ (3 s.f.)

c $P(K|J)=\dfrac{P(J\cap K)}{P(J)}=\dfrac{0.5}{0.6}=0.833$ (3 s.f.)

d $P(K|J)=0.833$ (3 s.f.), $P(K)=0.7$, so $P(K|J)\neq P(K)$$P(K|J)=0.833\ldots\neq P(K)=0.7$. So J and K are not independent.

Challenge

a The probability function must sum to 1. Therefore $k + 2k + 3k + 4k + 5k = 1 \Rightarrow 15k = 1 \Rightarrow k = \frac{1}{15}$

b $P(X = 5|X > 2) = \frac{P(X = 5)}{P(X > 2)} = \frac{\frac{5}{15}}{1 - P(X = 1 \cup X = 2)} = \frac{\frac{5}{15}}{1 - \frac{3}{15}} = \frac{\frac{5}{15}}{\frac{12}{15}} = \frac{5}{12}$

c

$$P(X \text{ is odd}|X \text{ is prime}) = \frac{P(X \text{ is odd and prime})}{P(X \text{ is prime})} = \frac{P(X = 3 \cup X = 5)}{P(X = 2 \cup X = 3 \cup X = 5)} = \frac{\frac{3+5}{15}}{\frac{2+3+5}{15}} = \frac{8}{10} = \frac{4}{5}$$