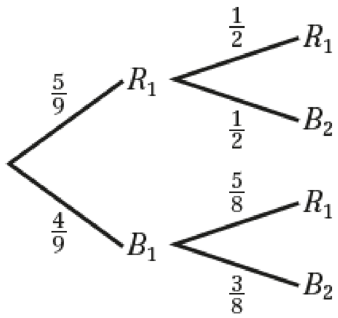
Conditional probability 2E

1 a When the first token removed is red, there are 8 tokens remaining in the bag, 4 red and 4 blue. When the first token removed is blue, there are 8 tokens remaining in the bag, 5 red and 3 blue.



b The answer can be read off from the tree diagram, following the lower branch (first blue) and then the red branch.

So $\text{P(second red}|\text{first blue)} = \frac{5}{8}$

c $$\text{P(first red}|\text{second blue)} = \frac{\text{P(first red and second blue)}}{\text{P(second blue)}} = \frac{\frac{5}{9}\times\frac{1}{2}}{\left(\frac{5}{9}\times\frac{1}{2}\right)+\left(\frac{4}{9}\times\frac{3}{8}\right)} = \frac{\frac{5}{18}}{\frac{32}{72}} = \frac{20}{32} = \frac{5}{8}$$

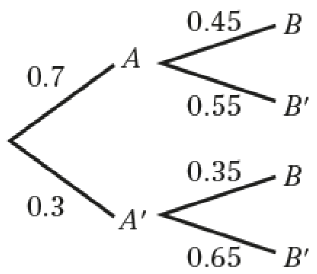
d $$\text{P(first blue}|\text{tokens different colours)} = \frac{\text{P(first blue and second red)}}{\text{P(first blue and second red)} + \text{P(first red and second blue)}}$$

$$= \frac{\frac{4}{9}\times\frac{5}{8}}{\left(\frac{4}{9}\times\frac{5}{8}\right)+\left(\frac{5}{9}\times\frac{1}{2}\right)} = \frac{\frac{20}{72}}{\frac{40}{72}} = \frac{20}{40} = \frac{1}{2}$$

e $$\text{P(tokens same colour}|\text{second token red)} = \frac{\text{P(first red and second red)}}{\text{P(first red and second red)} + \text{P(first blue and second red)}}$$

$$= \frac{\frac{5}{9}\times\frac{1}{2}}{\left(\frac{5}{9}\times\frac{1}{2}\right)+\left(\frac{4}{9}\times\frac{5}{8}\right)} = \frac{\frac{5}{18}}{\frac{40}{72}} = \frac{\frac{20}{72}}{\frac{40}{72}} = \frac{20}{40} = \frac{1}{2}$$

2 a $P(A) = 0.7 \Rightarrow P(A') = 1 - 0.7 = 0.3$
$P(B|A) = 0.45 \Rightarrow P(B'|A) = 1 - 0.45 = 0.55$
$P(B|A') = 0.35 \Rightarrow P(B'|A') = 1 - 0.35 = 0.65$
Therefore the completed tree diagram should be:



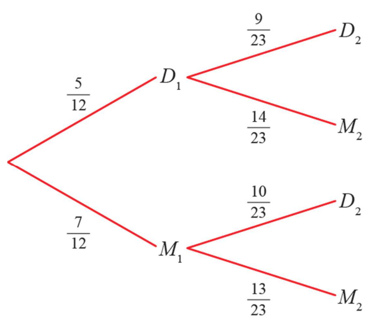
b i $P(A \cap B) = P(A) \times P(B|A) = 0.7 \times 0.45 = 0.315$

ii $P(A' \cap B') = P(A') \times P(B'|A') = 0.3 \times 0.65 = 0.195$

iii $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.315}{P(A \cap B) + P(A' \cap B)} = \dfrac{0.315}{0.315 + 0.3 \times 0.35} = \dfrac{0.315}{0.42} = 0.75$

3 a There are 10 dark chocolates in a box of 24, meaning the probability of choosing a dark chocolate is $\frac{10}{24} = \frac{5}{12}$. Similarly there are 14 milk chocolates out of the 24, and so the probability of choosing a dark chocolate is $\frac{14}{24} = \frac{7}{12}$.

Once Linda has eaten one chocolate, there are 23 chocolates left in the box. If the first chocolate she ate was a dark one, the probability of choosing another dark chocolate is $\frac{9}{23}$, and the probability of choosing a milk chocolate is $\frac{14}{23}$. If the first chocolate she ate was a milk one, the probability of a dark chocolate is $\frac{10}{23}$, and the probability of choosing another milk chocolate is $\frac{13}{23}$.



b $P(\text{dark and dark}) = \dfrac{5}{12} \times \dfrac{9}{23} = \dfrac{15}{92} = 0.163$ (3 s.f.)

3 c $P(\text{one dark and one milk}) = P(D_1 \cap M_2) + P(M_1 \cap D_2)$

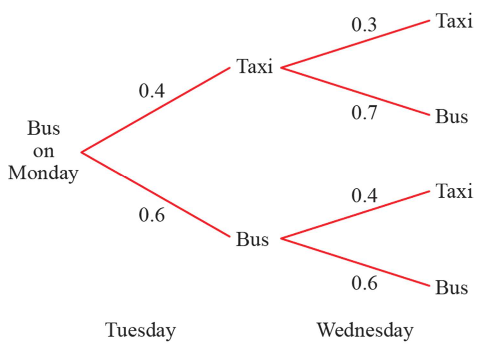
$$= \frac{5}{12} \times \frac{14}{23} + \frac{7}{12} \times \frac{10}{23}$$
$$= \frac{70}{276} + \frac{70}{276} = \frac{140}{276} = \frac{35}{69} = 0.507 \text{ (3 s.f.)}$$

d $\text{P(dark and dark|at least one dark)} = \dfrac{\text{P(dark and dark)}}{\text{P(at least one dark)}}$

$$= \frac{\text{P}(D_1 \cap D_2)}{\text{P}(D_1 \cap D_2) + \text{P}(D_1 \cap M_2) + \text{P}(M_1 \cap D_2)}$$

$$= \frac{\frac{5}{12} \times \frac{9}{23}}{\frac{5}{12} \times \frac{9}{23} + \frac{5}{12} \times \frac{14}{23} + \frac{7}{12} \times \frac{10}{23}} = \frac{\frac{45}{276}}{\frac{185}{276}} = \frac{45}{185} = \frac{9}{37} = 0.243 \text{ (3 s.f.)}$$

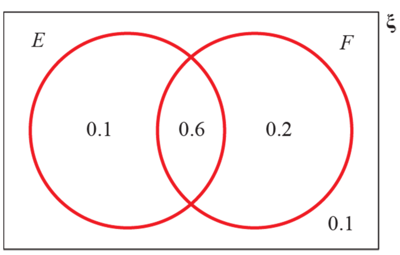
4 Use the information in the question to produce a tree diagram covering Jean's possible travel arrangements on Tuesday and Wednesday as follows:



Now sum the probabilities of Jean taking a taxi to work on Wednesday

$$\text{P(taxi on Wednesday)} = 0.4 \times 0.3 + 0.6 \times 0.4$$
$$= 0.12 + 0.24$$
$$= 0.36$$

5 Represent the information as a tree diagram. The coins are chosen at random, so there is a probability of $\frac{1}{2}$ of choosing each coin.

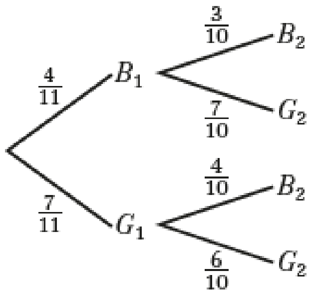


a $\text{P(head)} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 0.25$

5 b $\text{P(Fair|tail)} = \dfrac{\text{P(Fair and Tail)}}{\text{P(Tail)}}$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1} = \frac{1}{3} = 0.333 \text{ (3 s.f.)}$$

6 a Since the first ball selected is not replaced, there are 10 balls in the bag when the second ball is selected.



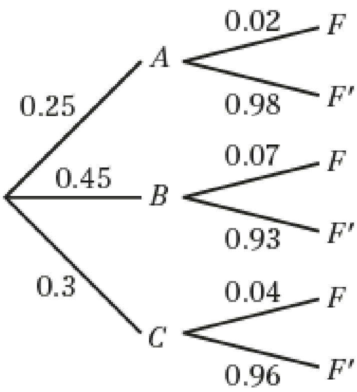
b $\text{P(second ball is green)} = \text{P}(B_1 \cap G_2) + \text{P}(G_1 \cap G_2)$

$$= \left(\frac{4}{11} \times \frac{7}{10}\right) + \left(\frac{7}{11} \times \frac{6}{10}\right) = \frac{28+42}{110} = \frac{7}{11} = 0.636 \text{ (3 s.f.)}$$

c $\text{P(both balls are green|second ball is green)} = \dfrac{\text{P(both balls are green)}}{\text{P(second ball is green)}}$

$$= \frac{\frac{7}{11} \times \frac{6}{10}}{\frac{70}{110}} = \frac{\frac{42}{110}}{\frac{70}{110}} = \frac{42}{70} = \frac{3}{5} = 0.6$$

7 a The probability of the sheet coming from *A, B* or *C* is given in the question. In each case, the probability that a sheet is flawed immediately provides the probability that it is not flawed since the two probabilities must sum to 1. Therefore the completed tree diagram is:



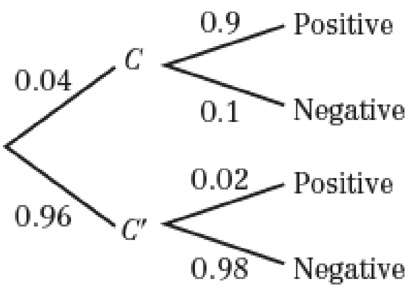
b i $\text{P(produced by } B \cap \text{flawed)} = 0.45 \times 0.07 = 0.0315$

7 b ii P(flawed) can be found by summing $\text{P(produced by } A \cap \text{flawed)}$, $\text{P(produced by } B \cap \text{flawed)}$ and $\text{P(produced by } C \cap \text{flawed)}$. Therefore

$$\text{P(flawed)} = 0.25 \times 0.02 + 0.0315 + 0.3 \times 0.04 = 0.0485$$

c $\text{P(produced by } A|\text{flawed)} = \dfrac{\text{P(produced by } A \cap \text{flawed)}}{\text{P(flawed)}} = \dfrac{0.25 \times 0.02}{0.0485} = 0.103 \text{ (3 s.f.)}$

8 a The reliability of the test depends on whether the person has the condition (C) or not (C').



b $\text{P(tests negative)} = \text{P}(C \cap \text{tests negative}) + \text{P}(C' \cap \text{tests negative})$
$= 0.04 \times 0.1 + 0.96 \times 0.98 = 0.9448 = 0.945 \text{ (3 s.f.)}$

c $\text{P(has condition}|\text{tests negative)} = \dfrac{\text{P}(C \cap \text{tests negative})}{\text{P(tests negative)}}$
$= \dfrac{0.04 \times 0.1}{0.9448} = 0.00423 \text{ (3 s.f.)}$

d From the data in the question, the test fails to find 10% of the people with the condition (since it has a 0.1 chance of producing a negative result when a person has the condition).

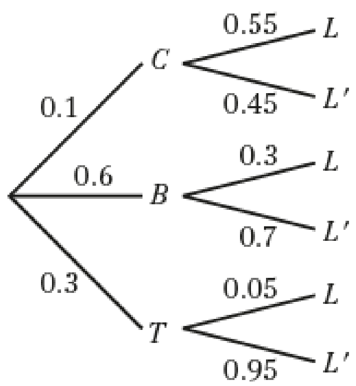
Consider also false positives, the case of a person who does not have the condition returning a positive result.

$\text{P(does not have the condition}|\text{tests positive)} = \dfrac{\text{P}(C' \cap \text{tests positive})}{\text{P(tests positive)}} = \dfrac{0.96 \times 0.02}{1 - 0.9448} = 0.348 \text{ (3 s.f.)}$

So over one third of the positive tests are false positives.

This means that if the test was used on the entire population, 10% of the people with the condition would not be identified and over one third of the people with a positive result would actually not have the condition.

9 a Since the probabilities of being late are given, the probabilities for being on time (i.e. not late) for each type of transport are known, sine the probabilities must sum to 1. Therefore the completed tree diagram should be as follows:



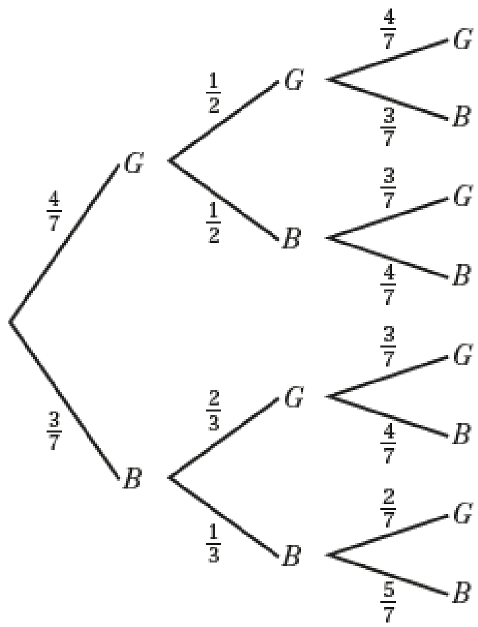
b i $\text{P(Bill travels by train and is late)} = 0.3 \times 0.05 = 0.015$

ii To find P(Bill is late), sum P(Bill travels by car and is late), P(Bill travels by bus and is late) and P(Bill travels by train and is late).

$\text{P(Bill is late)} = 0.1 \times 0.55 + 0.6 \times 0.3 + 0.3 \times 0.05 = 0.25$

c $\text{P(Bill travels by bus or train|Bill is late)} = \dfrac{0.6 \times 0.3 + 0.3 \times 0.05}{0.25} = 0.78$

10 a The two counters being drawn from box *A* can be modelled using a tree diagram. In each case, the number of counters of each colour in box *B* is then known, and so the third set of branches can be labelled to represent the drawing of the counter from box *B*. Therefore the completed tree diagram should be:



b $\text{P}(C) = \text{P}(GG) + \text{P}(BB) = \left(\dfrac{4}{7} \times \dfrac{1}{2}\right) + \left(\dfrac{3}{7} \times \dfrac{1}{3}\right) = \dfrac{2}{7} + \dfrac{1}{7} = \dfrac{3}{7}$

10 c $P(D) = P(GGB) + P(GBB) + P(BGB) + P(BBB)$

$$= \left(\frac{4}{7} \times \frac{1}{2} \times \frac{3}{7}\right) + \left(\frac{4}{7} \times \frac{1}{2} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{2}{3} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{1}{3} \times \frac{5}{7}\right)$$

$$= \frac{6}{49} + \frac{8}{49} + \frac{8}{49} + \frac{5}{49} = \frac{27}{49}$$

d The calculation will be similar to that for P(D), but with the first and second counters being the same colour.

$$P(C \cap D) = P(GGB) + P(BBB) = \left(\frac{4}{7} \times \frac{1}{2} \times \frac{3}{7}\right) + \left(\frac{3}{7} \times \frac{1}{3} \times \frac{5}{7}\right) = \frac{6}{49} + \frac{5}{49} = \frac{11}{49}$$

e Use the addition formula

$$P(C \cup D) = P(C) + P(D) - P(C \cap D) = \frac{3}{7} + \frac{27}{49} - \frac{11}{49} = \frac{21 + 27 - 11}{49} = \frac{37}{49}$$

f The required probability is:

$$\frac{P(GGG)}{P(GGG) + P(BBB)} = \frac{\frac{4}{7} \times \frac{1}{2} \times \frac{4}{7}}{\left(\frac{4}{7} \times \frac{1}{2} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{1}{3} \times \frac{5}{7}\right)} = \frac{\frac{8}{49}}{\frac{8}{49} + \frac{5}{49}} = \frac{8}{13} = 0.615 \text{ (3 s.f.)}$$

11 She has not taken into account the fact that after the first jelly bean is selected, there are only 9 jelly beans left in the box. So if the first jelly bean selected is sweet, the probability that the second bean is sweet is $\frac{6}{9}$ not $\frac{7}{10}$.

This is the correct solution.

$$P(\text{both jelly beans are sweet}) = \frac{7}{10} \times \frac{6}{9} = \frac{7}{15}$$

$$P(\text{at least one jelly bean is sweet}) = 1 - P(\text{neither jelly bean is sweet}) = 1 - \left(\frac{3}{10} \times \frac{2}{9}\right) = \frac{14}{15}$$

$$P(\text{both are sweet given at least one is sweet}) = \frac{\frac{7}{15}}{\frac{14}{15}} = \frac{7}{14} = 0.5$$

The correct answer is therefore 0.5.