Further kinematics 8E

1 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int (6t^2\,\mathbf{i} + (8 - 4t^3)\,\mathbf{j})\mathrm{d}t$

$= 2t^3\,\mathbf{i} + (8t - t^4)\mathbf{j} + C$

When $t = 0,\ \mathbf{v} = 0\,\mathbf{i} + 0\,\mathbf{j}$

$0\,\mathbf{i} + 0\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + C \Rightarrow C = 0\,\mathbf{i} + 0\,\mathbf{j}$

Hence

$\mathbf{v} = 2t^3\,\mathbf{i} + (8t - t^4)\,\mathbf{j}$

When $t = 2$

$\mathbf{v} = 16\,\mathbf{i} + (8 \times 2 - 2^4)\,\mathbf{j} = 16\,\mathbf{i}$

The velocity of P when $t = 2$ is $16\,\mathbf{i}\ \mathrm{m\,s^{-1}}$

b $\mathbf{r} = \int \mathbf{v}\mathrm{d}t = \int (2t^3\,\mathbf{i} + (8t - t^4)\,\mathbf{j})\mathrm{d}t$

$= \frac{1}{2}t^4\,\mathbf{i} + \left(4t^2 - \frac{1}{5}t^5\right)\mathbf{j} + D$

When $t = 0,\ \mathbf{r} = 0\,\mathbf{i} + 0\,\mathbf{j}$

$0\,\mathbf{i} + 0\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + D \Rightarrow D = 0\,\mathbf{i} + 0\,\mathbf{j}$

Hence

$$\mathbf{r} = \frac{t^4}{2}\mathbf{i} + \left(4t^2 - \frac{t^5}{5}\right)\mathbf{j}$$

When $t = 4$

$$\mathbf{r} = \frac{4^4}{2}\mathbf{i} + \left(4 \times 4^2 - \frac{4^5}{5}\right)\mathbf{j} = 128\,\mathbf{i} - 140.8\,\mathbf{j}$$

The position vector of P when $t = 4$ is $(128\,\mathbf{i} - 140.8\,\mathbf{j})\,\mathrm{m}$

2 a $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int ((3t^2 + 2)\,\mathbf{i} + (6t - 4)\,\mathbf{j})\,\mathrm{d}t$

$= (t^3 + 2t)\,\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j} + A$

When $t = 2,\ \mathbf{v} = 9\,\mathbf{j}$

$9\,\mathbf{j} = 12\,\mathbf{i} + 4\,\mathbf{j} + A \Rightarrow A = -12\,\mathbf{i} + 5\,\mathbf{j}$

Hence

$\mathbf{r} = (t^3 + 2t - 12)\,\mathbf{i} + \left(3t^2 - 4t + 5\right)\mathbf{j}$

When $t = 0$

$\mathbf{r} = -12\,\mathbf{i} + 5\,\mathbf{j}$

$|\,\mathbf{r}\,|^2 = (-12)^2 + 5^2 = 169 \Rightarrow |\,\mathbf{r}\,| = \sqrt{169} = 13$

The distance of P from O when $t = 0$ is 13 m.

2 b When P is moving parallel to $\mathbf{i}$, $\mathbf{v}$ has no $\mathbf{j}$ component.

$\Rightarrow 6t - 4 = 0$

$\Rightarrow t = \frac{2}{3}$

$\mathbf{a} = \dot{\mathbf{v}} = 6t\,\mathbf{i} + 6\,\mathbf{j}$

When $t = \frac{2}{3}$ s, $\mathbf{a} = 4\,\mathbf{i} + 6\,\mathbf{j}$

The acceleration of P at the instant when it is moving parallel to the vector $\mathbf{i}$ is $(4\,\mathbf{i} + 6\,\mathbf{j})\,\mathrm{m\,s^{-2}}$

3 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \left((2t-4)\mathbf{i} + 6\sin t\mathbf{j}\right)\,\mathrm{d}t$

$= \left(t^2 - 4t\right)\mathbf{i} - 6\cos t\mathbf{j} + c$

When $t = \frac{\pi}{2}$ s, $\mathbf{v} = 0$ ms^{-1}, so

$$0 = \left(\frac{\pi^2}{4} - \frac{4\pi}{2}\right)\mathbf{i} - 0\mathbf{j} + c$$

$$c = \left(2\pi - \frac{\pi^2}{4}\right)\mathbf{i}$$

The velocity of the particle is given by $\left[\left(t^2 - 4t + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 6\cos t\mathbf{j}\right]$ ms^{-1}

b When $t = \frac{3\pi}{2}$ s,

$$\mathbf{v} = \left(\frac{9\pi^2}{4} - \frac{12\pi}{2} + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 0\mathbf{j}$$

$$\mathbf{v} = \left(\frac{8\pi^2}{4} - 6\pi + 2\pi\right)\mathbf{i}$$

$$\mathbf{v} = \left(2\pi^2 - 4\pi\right)\mathbf{i}$$

Since the velocity only has an $\mathbf{i}$ component when $t = \frac{3\pi}{2}$ s, this is also the speed.

The speed of P at $\frac{3\pi}{2}$ s is $\left(2\pi^2 - 4\pi\right)$ ms^{-1}

4 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int ((5t-3)\,\mathbf{i} + (8-t)\,\mathbf{j})\,\mathrm{d}t$

$= \left(\frac{5}{2}t^2 - 3t\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2\right)\mathbf{j} + C$

When $t = 0$, $\mathbf{v} = 2\,\mathbf{i} - 5\,\mathbf{j}$

$2\,\mathbf{i} - 5\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + C \Rightarrow C = 2\,\mathbf{i} - 5\,\mathbf{j}$

Hence

$$\mathbf{v} = \left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}$$

The velocity of P after t seconds is $\left(\left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}\right)$ m s^{-1}

4 b P is moving parallel to $\mathbf{i}-\mathbf{j}$ when, in the expression giving the velocity of P

$(\text{coefficient of } \mathbf{i} \text{ component}) = -1\times(\text{coefficient of j component})$

$$\left(\frac{5}{2}t^2-3t+2\right)=-\left(8t-\frac{1}{2}t^2-5\right)$$

$$\frac{5}{2}t^2-3t+2=-8t+\frac{1}{2}t^2+5$$

$$2t^2+5t-3=0$$

$$(2t-1)(t+3)=0$$

Hence,

$$t=\frac{1}{2},\ -3$$

As $t\geq 0$, $t=\frac{1}{2}$

c When $t=\frac{1}{2}$

$$\mathbf{v}=\left(\frac{5}{8}-\frac{3}{2}+2\right)\mathbf{i}+\left(4-\frac{1}{8}-5\right)\mathbf{j}$$

$$=\frac{9}{8}\mathbf{i}-\frac{9}{8}\mathbf{j}$$

$$|\mathbf{v}|^2=\left(\frac{9}{8}\right)^2+\left(\frac{9}{8}\right)^2=2\times\left(\frac{9}{8}\right)^2$$

$$\Rightarrow|\mathbf{v}|=\frac{9\sqrt{2}}{8}$$

The speed of P when it is moving parallel to $\mathbf{i}-\mathbf{j}$ is $\frac{9\sqrt{2}}{8}$ m s^{-1}

5 a $\mathbf{v}=\int\mathbf{a}\,\mathrm{d}t=\int(2\mathbf{i}-2t\,\mathbf{j})\,\mathrm{d}t$

$$=2t\,\mathbf{i}-t^2\mathbf{j}+A$$

When $t=0$, $\mathbf{v}=2\mathbf{j}$

$$2\mathbf{j}=0\mathbf{i}+0\mathbf{j}+A\Rightarrow A=2\mathbf{j}$$

Hence

$$\mathbf{v}=2t\,\mathbf{i}+(2-t^2)\,\mathbf{j}$$

Let the position vector of P at time t seconds be $\mathbf{p}$ m.

$$\mathbf{p}=\int\mathbf{v}\,\mathrm{d}t=\int 2t\,\mathbf{i}+(2-t^2)\,\mathbf{j}$$

$$=t^2\,\mathbf{i}+\left(2t-\frac{1}{3}t^3\right)\mathbf{j}+B$$

When $t=0$, $\mathbf{v}=6\mathbf{i}$

$$6\mathbf{i}=0\mathbf{i}+0\mathbf{j}+B\Rightarrow B=6\mathbf{i}$$

Hence

$$\mathbf{p}=(t^2+6)\mathbf{i}+\left(2t-\frac{1}{3}t^3\right)\mathbf{j}$$

The position vector of P at time t seconds is $\left((t^2+6)\mathbf{i}+\left(2t-\frac{1}{3}t^3\right)\mathbf{j}\right)$ m

5 b Let the position vector of Q at time t seconds be $\mathbf{q}$ m.

$$\mathbf{q} = \int \mathbf{v}\,dt = \int ((3t^2 - 4)\mathbf{i} - 2t\,\mathbf{j})dt$$
$$= (t^3 - 4t)\mathbf{i} - t^2\,\mathbf{j} + C$$

From part **a**, when $t = 3$

$$\mathbf{p} = (3^2 + 6)\mathbf{i} + \left(2 \times 3 - \frac{3^3}{3}\right)\mathbf{j} = 15\mathbf{i} - 3\mathbf{j}$$

As the particles collide when $t = 3$, $\mathbf{q}(3) = \mathbf{p}(3)$

$$\mathbf{p}(3) = \mathbf{q}(3)$$
$$15\mathbf{i} - 3\mathbf{j} = (3^3 - 4 \times 3)\mathbf{i} - 3^2\,\mathbf{j} + C$$
$$15\mathbf{i} - 3\mathbf{j} = 15\mathbf{i} - 9\mathbf{j} + C$$
$$C = 6\mathbf{j}$$

Hence,

$$\mathbf{q} = (t^3 - 4t)\mathbf{i} + (6 - t^2)\mathbf{j}$$

When $t = 0$, $\mathbf{q} = 6\mathbf{j}$

The position vector of Q at time $t = 0$ is $6\mathbf{j}$ m

6 a

$$\mathbf{v} = \int \mathbf{a}\,dt = \int ((4t - 3)\mathbf{i} - 6t^2 t\,\mathbf{j})dt$$
$$= (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j} + A$$

When $t = 0$, $\mathbf{v} = 0$

$$0 = 0\mathbf{i} + 0\mathbf{j} + A \Rightarrow A = 0$$
$$\mathbf{v} = (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j}$$

When $t = \frac{1}{2}$

$$\mathbf{v} = \left(2\left(\frac{1}{2}\right)^2 - 3 \times \frac{1}{2}\right)\mathbf{i} - 2\left(\frac{1}{2}\right)^3\mathbf{j}$$
$$= -\mathbf{i} - \frac{1}{4}\mathbf{j}$$

The velocity of P when $t = \frac{1}{2}$ is $\left(-\mathbf{i} - \frac{1}{4}\mathbf{j}\right)\text{m s}^{-1}$

b

$$\mathbf{r} = \int \mathbf{v}\,dt = \int ((2t^2 - 3t)\mathbf{i} - 2t^3\,\mathbf{j})dt$$
$$= \left(\frac{2}{3}t^3 - \frac{3}{2}t^2\right)\mathbf{i} - \frac{1}{2}t^4\,\mathbf{j} + B$$

When $t = 0$, $\mathbf{r} = 4\mathbf{i} - 6\mathbf{j}$

$$4\mathbf{i} - 6\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + B \Rightarrow B = 4\mathbf{i} - 6\mathbf{j}$$
$$r = \left(\frac{2}{3}t^3 - \frac{3}{2}t^2 + 4\right)\mathbf{i} - \left(\frac{1}{2}t^4 + 6\right)\mathbf{j}$$

When $t = 6$

$$\mathbf{r} = (144 - 54 + 4)\mathbf{i} - (648 + 6)\mathbf{j} = 94\mathbf{i} - 654\mathbf{j}$$

The position vector of P when $t = 6$ is $(94\mathbf{i} - 654\mathbf{j})$ m

7 a $\mathbf{v} = \int \mathbf{a}\,dt = \int ((8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j})dt$

$= (2t^4 - 3t^2)\mathbf{i} + (4t^2 - 3t)\mathbf{j} + C$

When $t = 2$, $\mathbf{v} = 16\mathbf{i} + 3\mathbf{j}$

$16\mathbf{i} + 3\mathbf{j} = 20\mathbf{i} + 10\mathbf{j} + C \Rightarrow C = -4\mathbf{i} - 7\mathbf{j}$

$\mathbf{v} = (2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}$

The velocity of P after t seconds is $\left((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}\right)$ m s^{-1}

b When P is moving parallel to **i**, the **j** component of the velocity is zero.

$4t^2 - 3t - 7 = 0$

$(t + 1)(4t - 7) = 0$

$t \geq 0 \Rightarrow t = \frac{7}{4}$ s

8 a $\mathbf{r}_P = \int \mathbf{v}_P\,dt = \int \left((4t - 3)\mathbf{i} + 4\mathbf{j}\right)dt$

$= \left(2t^2 - 3t\right)\mathbf{i} + 4t\mathbf{j} + c$

When $t = 0$ s, $\mathbf{r}_P = (\mathbf{i} + 2\mathbf{j})$ m

$\mathbf{i} + 2\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = \mathbf{i} + 2\mathbf{j}$

The position of P at time t is given by $\left(\left(2t^2 - 3t + 1\right)\mathbf{i} + \left(4t + 2\right)\mathbf{j}\right)$m.

b i $\mathbf{r}_Q = \int \mathbf{v}_Q\,dt = \int 5\mathbf{i} + k\mathbf{j}\,dt$

$= 5t\mathbf{i} + kt\mathbf{j} + c$

When $t = 0$ s, $\mathbf{r} = (11\mathbf{i} + 5\mathbf{j})$ m

$11\mathbf{i} + 5\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = 11\mathbf{i} + 5\mathbf{j}$

$\mathbf{r}_Q = (5t + 11)\mathbf{i} + (kt + 5)\mathbf{j}$

When the particles collide, their position vectors are identical, so:

$$\mathbf{r}_P = \mathbf{r}_Q$$

$\left(2t^2 - 3t + 1\right)\mathbf{i} + \left(4t + 2\right)\mathbf{j} = (5t + 11)\mathbf{i} + (kt + 5)\mathbf{j}$

Considering the coefficients of **i**:

$2t^2 - 3t + 1 = 5t + 11$

$2t^2 - 8t - 10 = 0$

$t^2 - 4t - 5 = 0$

$(t - 5)(t + 1) = 0$

The negative root can be ignored, so the particles collide when $t = 5$ s

Equating the coefficients of **j** when $t = 5$ s:

$20 + 2 = 5k + 5$

$k = \frac{22 - 5}{5} = 3.4$

The value of k is 3.4

8 b ii Substituting $k = 3.4$ and $t = 5$ into equation for $\mathbf{r}_Q$:

$$\mathbf{r}_Q = (25+11)\mathbf{i} + ((5\times 3.4)+5)\mathbf{j}$$
$$= 36\mathbf{i} + 22\mathbf{j}$$

The position vector of the points where the particles meet is $(36\mathbf{i}+22\mathbf{j})$ m.

Challenge

$\mathbf{v} = (3t\cos t\mathbf{i} + 5t\mathbf{j})$ ms^{-1}, $\mathbf{r_o} = (4\mathbf{i}+\mathbf{j})$m, $t = 0$ s

$$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int (3t\cos t\mathbf{i} + 5t\mathbf{j})\,\mathrm{d}t$$

To evaluate $\int t\cos t\,\mathrm{d}t$, let $u = t$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \cos t$

Then $\frac{\mathrm{d}u}{\mathrm{d}t} = 1$ and $v = \sin t$

Using integration by parts, $\int t\cos t\,\mathrm{d}t = t\sin t - \int \sin t\,\mathrm{d}t$

$$= t\sin t + \cos t \qquad \textbf{(1)}$$

$$\mathbf{r} = \int (3t\cos t\mathbf{i} + 5t\mathbf{j})\,\mathrm{d}t$$
$$= \left(3\int t\cos t\,\mathrm{d}t\right)\mathbf{i} + \left(5\int t\,\mathrm{d}t\right)\mathbf{j}$$
$$= 3(t\sin t + \cos t)\mathbf{i} + \frac{5t^2}{2}\mathbf{j} + c \qquad \text{(using } \textbf{(1)}\text{)}$$

When $t = 0$ s, $\mathbf{r} = (4\mathbf{i}+\mathbf{j})$m

$$4\mathbf{i}+\mathbf{j} = 3(0+1)\mathbf{i} + 0\mathbf{j} + c$$
$$c = \mathbf{i}+\mathbf{j}$$

Hence, $\mathbf{r} = (3(t\sin t + \cos t)+1)\mathbf{i} + \left(\frac{5t^2}{2}+1\right)\mathbf{j}$

When $t = \frac{\pi}{2}$,

$$\mathbf{r} = \left(3\left(\frac{\pi}{2}\sin\frac{\pi}{2}+0\right)+1\right)\mathbf{i} + \left(\frac{5\pi^2}{2\times 4}+1\right)\mathbf{j}$$
$$\mathbf{r} = \left(\frac{3\pi}{2}+1\right)\mathbf{i} + \left(\frac{5\pi^2}{8}+1\right)\mathbf{j}$$

The position of P at time $t = \frac{\pi}{2}$ s is $\left(\left(\frac{3\pi}{2}+1\right)\mathbf{i} + \left(\frac{5\pi^2}{8}+1\right)\mathbf{j}\right)$ m relative to O.