Further kinematics Mixed exercise 8

1 $\mathbf{u}=0,\ t=5,\ \mathbf{v}=6\mathbf{i}-8\mathbf{j},\ \mathbf{a}=?$

Using $\mathbf{v}=\mathbf{u}+\mathbf{a}t$,

$(6\mathbf{i}-8)=\mathbf{a}\times 5,$

$$\mathbf{a}=\frac{1}{5}(6\mathbf{i}-8\mathbf{j})$$

Using $\mathbf{F}=m\mathbf{a}$,

$$\mathbf{F}=4\times\frac{1}{5}(6\mathbf{i}-8\mathbf{j})$$
$$=4.8\mathbf{i}-6.4\mathbf{j}$$

2 Using $\mathbf{F}=m\mathbf{a}$,

$(2\mathbf{i}-\mathbf{j})=2\mathbf{a},$

$$\mathbf{a}=\mathbf{i}-\frac{1}{2}\mathbf{j}$$

$\mathbf{u}=\mathbf{i}+3\mathbf{j},\ t=3,\ \mathbf{a}=\mathbf{i}-\frac{1}{2}\mathbf{j},\ \mathbf{s}=?$

Using $\mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$,

$$\mathbf{s}=(\mathbf{i}+3\mathbf{j})\times 3+\frac{1}{2}\left(\mathbf{i}-\frac{1}{2}\mathbf{j}\right)\times 3^2$$
$$=3\mathbf{i}+9\mathbf{j}+\frac{9}{2}\mathbf{i}-\frac{9}{4}\mathbf{j}$$
$$=\frac{15}{2}\mathbf{i}+\frac{27}{4}\mathbf{j}$$

$$\text{distance}=\sqrt{\left(\frac{15}{2}\right)^2+\left(\frac{27}{4}\right)^2}$$
$$=\sqrt{56.25+45.5625}$$
$$=\sqrt{101.8125}$$
$$=10.1\text{ m (3 s.f.)}$$

3 a Using $\mathbf{r}=\mathbf{r}_0+\mathbf{v}t$,

$$\mathbf{r}=-500\mathbf{j}+(2\mathbf{i}+3\mathbf{j})\times t$$
$$=-500\mathbf{j}+2t\mathbf{i}+3t\mathbf{j}$$
$$=2t\mathbf{i}+(-500+3t)\mathbf{j}$$

3 b $5 \text{ minutes} = 5 \times 60 \text{ seconds}$
$= 300 \text{ seconds}$

At 2.05 pm, the dinghy has position:

$\mathbf{r} = 2 \times 300\,\mathbf{i} + (-500 + 3 \times 300)\,\mathbf{j}$
$= 600\,\mathbf{i} + 400\,\mathbf{j}$

$\text{distance} = \sqrt{600^2 + 400^2}$
$= \sqrt{360\,000 + 160\,000}$
$= \sqrt{520\,000}$
$= 721\text{m} \quad (3 \text{ s.f.})$

4 a Using $\mathbf{r}_A = \mathbf{r}_{A_0} + \mathbf{v}_A t$ for **A**,

$\mathbf{r}_A = (\mathbf{i} + 3\,\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \times t$
$= (1 + 2t)\mathbf{i} + (3 - t)\mathbf{j}$

Using $\mathbf{r}_B = \mathbf{r}_{B_0} + \mathbf{v}_B t$ for **B**,

$\mathbf{r}_B = (5\mathbf{i} - 2\,\mathbf{j}) + (-\mathbf{i} + 4\mathbf{j}) \times t$
$= (5 - t)\mathbf{i} + (-2 + 4t)\mathbf{j}$

b $\mathbf{r}_{\overline{AB}} = \mathbf{r}_B - \mathbf{r}_A$
$= \big((5 - t)\,\mathbf{i} + (-2 + 4t)\,\mathbf{j}\big) - \big((1 + 2t)\,\mathbf{i} + (3 - t)\,\mathbf{j}\big)$
$= (5 - t - 1 - 2t)\,\mathbf{i} + (-2 + 4t - 3 + t)\,\mathbf{j}$
$= (4 - 3t)\,\mathbf{i} + (-5 + 5t)\,\mathbf{j}$

c If A and B collide, the vector ***AB*** would be zero, so $4 - 3t = 0$ and $-5 + 5t = 0$, but these two equations are not consistent $(t = 1 \text{ and } t \neq 1)$, so vector ***AB*** can never be zero and A and B will not collide.

d At 10 am, $t = 2$:

$\mathbf{r}_{\overline{AB}} = (4 - 3 \times 2)\,\mathbf{i} + (-5 + 5 \times 2)\,\mathbf{j}$
$= -2\,\mathbf{i} + 5\,\mathbf{j}$

$\text{Distance} = \sqrt{(-2)^2 + 5^2}$
$= \sqrt{29}$
$= 5.39\,\text{km}$

5 Let x be the horizontal distance between O and S, and y be the vertical distance between O and S.
$\mathbf{u} = 8\mathbf{i} + 10\mathbf{j},\ \mathbf{a} = -9.8\mathbf{j},\ t = 6,\ \mathbf{s} = x\mathbf{i} + y\mathbf{j}$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$x\mathbf{i} + y\mathbf{j} = (8\mathbf{i} + 10\mathbf{j}) \times 6 + \frac{1}{2}(-9.8\mathbf{j}) \times 36$$

a Equating **i** components:
$x = 48$
The horizontal distance between O and S is 48 m.

b Equating **j** components:
$y = 60 - 4.9 \times 36$
$= -116$
The vertical distance between O and S is 116 m (3 s.f.).

6 $\mathbf{u} = (p\mathbf{i} + q\mathbf{j})\,\text{ms}^{-1},\ \mathbf{r}_0 = 0.8\mathbf{j}\text{ m},\ \mathbf{a} = -9.8\mathbf{j}\text{ ms}^{-2};\ t = 4\text{ s},\ \mathbf{r} = 64\mathbf{i}\text{ m}$

a Combining $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ and $\mathbf{s} = \mathbf{r} - \mathbf{r}_0$ gives

$$\mathbf{r} - \mathbf{r}_0 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Using vector notation:

$$\begin{pmatrix} 64 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0.8 \end{pmatrix} = 4\begin{pmatrix} p \\ q \end{pmatrix} + \frac{4^2}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\begin{pmatrix} 64 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 4p \\ 4q - 78.4 \end{pmatrix}$$

Considering **i** components:
$64 = 4p$
$p = 16$
Considering **j** components:
$-0.8 = 4q - 78.4$
$4q = 78.4 - 0.8$
$q = 19.6 - 0.2 = 19.4$

The values of p and q are 16 and 19.4 respectively. $\mathbf{s} = \begin{pmatrix} 16 \\ 19.4 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4.9 \end{pmatrix} t^2$

b $|\mathbf{u}| = \sqrt{16^2 + 19.4^2} = 25.146...$
The initial speed of the ball is 25.1 ms^{-1}.

c $\tan\alpha = \frac{q}{p} = \frac{19.4}{16}$

The exact value of $\tan\alpha$ is $\frac{97}{80}$

6 d Use $\mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$ and $\mathbf{s}=\mathbf{r}-\mathbf{r}_0$ to find values of t for which $\mathbf{r}=x\mathbf{i}+5\mathbf{j}$:

$$\begin{pmatrix}x\\5\end{pmatrix}-\begin{pmatrix}x\\0.8\end{pmatrix}=\begin{pmatrix}16\\19.4\end{pmatrix}t+\begin{pmatrix}0\\-4.9\end{pmatrix}t^2$$

Considering **j** components:

$$5-0.8=19.4t-4.9t^2$$

$$4.9t^2-19.4t+4.2=0$$

Using the equation for the roots of a quadratic equation:

$$t=\frac{19.4\pm\sqrt{19.4^2-(4\times4.9\times4.2)}}{2\times4.9}$$

$t=3.7293...$ or $t=0.2298...$

The ball is above 5 m between these two times, i.e. for 3.7293… − 0.2298…= 3.50 s (3s.f.).

e To make the model more realistic, one should consider factors such as air resistance and how it is affected by the shape (especially the seam) and the spin of the ball.

7 a $\mathbf{r}=\int\mathbf{v}\,\mathrm{d}t=\int t\left(2t^2+14\right)^{\frac{1}{2}}\mathrm{d}t$

$$=\frac{2}{3\times2\times2}\left(2t^2+14\right)^{\frac{3}{2}}+c$$

$$=\frac{1}{6}\left(2t^2+14\right)^{\frac{3}{2}}+c$$

$\mathbf{r}=0$ when $t=0$, hence

$$0=\frac{1}{6}\left(0+14\right)^{\frac{3}{2}}+c$$

$$c=-8.73$$

$$\Rightarrow\mathbf{r}=\frac{1}{6}\left(2t^2+14\right)^{\frac{3}{2}}-8.73$$

At t = 5 s, $\mathbf{r}=\frac{1}{6}\left(50+14\right)^{\frac{3}{2}}+8.73=76.6$

At t = 5 s, the displacement of P from O is 76.6 m (3s.f.).

b $\mathbf{v}=\frac{1000}{t^2}$ ms^{-1}, t = 5 s, **r** = 76.6 m; t = 6 s, **r** = ?

$\mathbf{r}=\int\mathbf{v}\,\mathrm{d}t=\int1000t^{-2}\mathrm{d}t$

$$=-\frac{1000}{t}+c$$

Using that fact that at t = 5 s the position of the particle will be as given in part **a**:

$$76.6=\frac{-1000}{5}+c$$

$$c=76.6+200=276.6$$

$$\Rightarrow\mathbf{r}=\frac{-1000}{t}+276.6$$

At t = 6 s,

$$\mathbf{r}=\frac{-1000}{6}+276.6=109.9$$

At t = 6 s, the displacement of P from O is 110 m (3s.f.).

8 a $x = 2t + k(t+1)^{-1}$

$v = \frac{dx}{dt} = 2 - k(t+1)^{-2} = 2 - \frac{k}{(t+1)^2}$

When $t = 0$, $v = 6$

$6 = 2 - \frac{k}{1^2} \Rightarrow k = -4$

b With $k = -4$

$x = 2t - \frac{4}{t+1}$

When $t = 0$, $x = 0 - \frac{4}{0+1} = -4$

The distance of P from O when $t = 0$ is 4 m.

c $v = 2 - 4(t+1)^{-2}$

$a = \frac{dv}{dt} = 8(t+1)^{-3} = \frac{8}{(t+1)^3}$

When $t = 3$

$a = \frac{8}{4^3} = \frac{1}{8}$

$F = ma$

$= 0.4 \times \frac{1}{8} = 0.05$

The magnitude of **F** when $t = 3$ is 0.05.

9 a When $t = \frac{1}{2}$

$x = 0.6\cos\left(\frac{\pi}{3} \times \frac{1}{2}\right)$

$= 0.6\cos\frac{\pi}{6}$

$= 0.6 \times \frac{\sqrt{3}}{2} = 0.3\sqrt{3}$

The distance of B from O when $t = \frac{1}{2}$ is $0.3\sqrt{3}$ m.

b $v = \frac{dx}{dt} = -0.6 \times \frac{\pi}{3}\sin\left(\frac{\pi t}{3}\right)$

The smallest value at which $v = 0$ is given by

$\frac{\pi t}{3} = \pi \Rightarrow t = 3$ s.

9 c $a = \frac{dv}{dt} = -0.6\left(\frac{\pi}{3}\right)^2 \cos\left(\frac{\pi t}{3}\right)$

When $t = 1$

$a = -0.6\left(\frac{\pi}{3}\right)^2 \cos\left(\frac{\pi}{3}\right) = -0.3289...$

The magnitude of the acceleration of B when $t = 1$ is $0.329\,\text{m s}^{-2}$ (3 s.f.).

10 a $v = \frac{dx}{dt} = 4e^{-0.5t} - 2te^{-0.5t}$

$a = \frac{dv}{dt} = -2e^{-0.5t} - 2e^{-0.5t} + te^{-0.5t} = (t-4)e^{-0.5t}$

When $t = \ln 4$

$a = (\ln 4 - 4)e^{-0.5\ln 4}$

$= (\ln 2^2 - 4)e^{\ln 4^{-\frac{1}{2}}}$

$= (2\ln 2 - 4)e^{\ln\frac{1}{2}}$

$= \frac{1}{2}(2\ln 2 - 4)$

$= \ln 2 - 2$

The acceleration of S when $t = \ln 4$ is $(\ln 2 - 2)\,\text{m s}^{-2}$ in the direction of x increasing.

b For a maximum of x, $\frac{dx}{dt} = v = 0$

$v = (4 - 2t)e^{-0.5t} = 0 \Rightarrow t = 2$

When $t = 2$

$x = 4 \times 2e^{-0.5\times 2} = 8e^{-1}$

The greatest distance of S from O is $\frac{8}{e}$ m.

11 a $\mathbf{v}_P = \dot{\mathbf{r}}_P = 6t\mathbf{i} + 2\mathbf{j}$

$\mathbf{v}_Q = \dot{\mathbf{r}}_Q = \mathbf{i} + 3t\mathbf{j}$

$\frac{d}{dt}\left((t+6)\mathbf{i}\right) = 1\mathbf{i} = \mathbf{i}$

The velocity of P at time t seconds is $(6t\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$ and the velocity of Q is $(\mathbf{i} + 3t\mathbf{j})\,\text{m s}^{-1}$

b When $t = 2$

$\mathbf{v}_P = 12\mathbf{i} + 2\mathbf{j}$

$|\mathbf{v}_P|^2 = 12^2 + 2^2 = 148 \Rightarrow \mathbf{v}_P = \sqrt{148} = 12.165...$

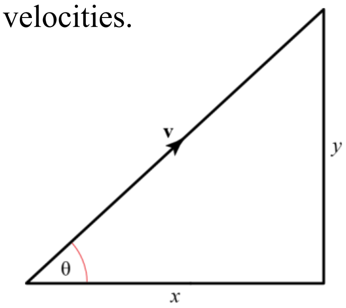
The speed of P when $t = 2$ is $12.2\,\text{m s}^{-1}$ (3 s.f.).

11 c When P is moving parallel to Q

$$\frac{2}{6t} = \frac{3t}{1}$$
$$\Rightarrow 18t^2 = 2$$
$$\Rightarrow t^2 = \frac{1}{9}$$
$$t \geqslant 0,\ t = \frac{1}{3}$$

When the particles are moving parallel to each other, the angle each makes with **i** is the same.

If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, $\tan\theta = \frac{y}{x}$ must be the same for both velocities.



d **i**-components

$$3t^2 + 4 = t + 6$$
$$3t^2 - t - 2 = 0$$
$$(t-1)(3t+2) = 0$$
$$t = 1,\ -\frac{2}{3}$$

j-components

$$2t - \frac{1}{2} = \frac{3t^2}{2}$$
$$3t^2 - 4t + 1 = 0$$
$$(t-1)(3t-1) = 0$$
$$t = 1,\ \frac{1}{3}$$

For the particles to collide, both the **i** and **j** components of their position vectors must be the same for the same value of t. The appropriate method is to equate the **i** components and solve the resulting quadratic, and then do the same for **j** components. If one of the roots of the quadratics is the same, then the particles collide.

1 is a common root of the equations and, hence, P and Q collide at the point with position vector $\left(7\mathbf{i} + \frac{3}{2}\mathbf{j}\right)$m.

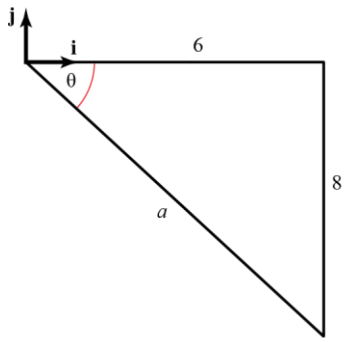
$t = 1$ can be substituted into either $\mathbf{r}_P$ or $\mathbf{r}_Q$ to find the position vector of the point where the particles collide.

12 a $\mathbf{v} = \dot{\mathbf{r}} = 6t\mathbf{i} - 8t\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} - 8\mathbf{j}$

Acceleration does not depend on t, hence the acceleration is constant.

12 b



$|\mathbf{a}|^2 = 6^2 + (-8)^2 = 100$

$\Rightarrow |\mathbf{a}| = 10$

The magnitude of the acceleration is $10\,\text{m s}^{-2}$

$\tan\theta = \frac{8}{6} \Rightarrow \theta = 53.1°$

The angle the acceleration makes with **j** is $90° + 53.1° = 143.1°$ (nearest 0.1°)

13 a $\mathbf{v} = \dot{\mathbf{r}} = -6\sin 3t\,\mathbf{i} - 6\cos 3t\,\mathbf{j}$

When $t = \frac{\pi}{6}$

$$\mathbf{v} = -6\sin\frac{\pi}{2}\mathbf{i} - 6\cos\frac{\pi}{2}\mathbf{j}$$
$$= -6\mathbf{i} - 0$$

The velocity of P when $t = \frac{\pi}{6}$ is $-6\mathbf{i}\ \text{m s}^{-1}$

b $\mathbf{a} = \dot{\mathbf{v}} = -18\cos 3t\mathbf{i} + 18\sin 3t\,\mathbf{j}$

$$|\mathbf{a}|^2 = \left(-18\cos 3t\right)^2 + \left(18\sin 3t\right)^2$$
$$= 18^2\left(\cos^2 3t + \sin^2 3t\right) = 18^2$$

$|\mathbf{a}| = 18$

The magnitude of the acceleration is $18\,\text{m s}^{-2}$, which is constant.

14 a $\mathbf{a} = \dot{\mathbf{v}} = 4c\mathbf{i} + 2(7-c)t\,\mathbf{j}$

$$\mathbf{F} = m\mathbf{a}$$
$$= 0.5\left(4c\,\mathbf{i} + 2(7-c)t\,\mathbf{j}\right)$$
$$= 2c\,\mathbf{i} + (7-c)t\,\mathbf{j}, \text{ as required}$$

b $t = 5 \Rightarrow \mathbf{F} = 2c\,\mathbf{i} + 5(7-c)\,\mathbf{j}$

$$|\mathbf{F}|^2 = 4c^2 + 25(7-c)^2 = 17^2$$
$$4c^2 + 1225 - 350c + 25c^2 = 289$$
$$29c^2 - 350c + 936 = 0$$
$$(c-4)(29c-234) = 0$$
$$c = 4,\ \frac{234}{29} \approx 8.07$$

15 a $\mathbf{v} = \int \mathbf{a}\mathrm{d}t = \int ((8t^3 - 6t)\,\mathbf{i} + (8t - 3)\,\mathbf{j})\mathrm{d}t$

$= (2t^4 - 3t^2)\,\mathbf{i} + (4t^2 - 3t)\mathbf{j} + C$

When $t = 2$, $\mathbf{v} = 16\,\mathbf{i} + 3\,\mathbf{j}$

$16\,\mathbf{i} + 3\,\mathbf{j} = 20\,\mathbf{i} + 10\,\mathbf{j} + C \Rightarrow C = -4\,\mathbf{i} - 7\,\mathbf{j}$

$\mathbf{v} = (2t^4 - 3t^2 - 4)\,\mathbf{i} + (4t^2 - 3t - 7)\,\mathbf{j}$

The velocity of P after t seconds is $\left((2t^4 - 3t^2 - 4)\,\mathbf{i} + (4t^2 - 3t - 7)\,\mathbf{j}\right)\mathrm{m\,s^{-1}}$

b When P is moving parallel to **i**, the **j** component of the velocity is zero.

$4t^2 - 3t - 7 = 0$

$(t+1)(4t-7) = 0$

$t \geqslant 0 \Rightarrow t = \frac{7}{4}$

16 $\mathbf{a} = \left(4t\mathbf{i} + 5t^{-\frac{1}{2}}\mathbf{j}\right)$ ms^{-2}, $t = 0$ s, $\mathbf{v} = 10\mathbf{i}$ ms^{-1}; $t = 5$ s, $|\mathbf{v}| = ?$

$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \left(4t\mathbf{i} + 5t^{-\frac{1}{2}}\mathbf{j}\right)\mathrm{d}t$

$= \frac{4t^2}{2}\mathbf{i} + \frac{5}{\frac{1}{2}}t^{\frac{1}{2}}\mathbf{j} + c$

$= 2t^2\mathbf{i} + 10t^{\frac{1}{2}}\mathbf{j} + c$

When $t = 0$ s, $\mathbf{v} = 10\,\mathbf{i}$ ms^{-1}

$10\mathbf{i} = 0\mathbf{i} - 0\mathbf{j} + c$

$c = 10\mathbf{i}$

$\Rightarrow \mathbf{v} = (2t^2 + 10)\mathbf{i} + 10t^{\frac{1}{2}}\mathbf{j}$

At $t = 5$ s,

$\mathbf{v} = (50 + 10)\mathbf{i} + 10\sqrt{5}\mathbf{j}$

$|\mathbf{v}| = \sqrt{60^2 + \left(10\sqrt{5}\right)^2} = \sqrt{4100}$

$|\mathbf{v}| = 10\sqrt{41}$

At $t = 5$ s, the speed of the ball is $10\sqrt{41}$ ms^{-1}.

17 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int 2t\,\mathbf{i} + 3\mathbf{j}\,\mathrm{d}t$

$= t^2\mathbf{i} + 3t\mathbf{j} + c$

When $t = 0$ s, $v = 3\mathbf{i} + 13\mathbf{j}$

$3\mathbf{i} + 13\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = 3\mathbf{i} + 13\mathbf{j}$

$\mathbf{v} = \left(t^2 + 3\right)\mathbf{i} + \left(3t + 13\right)\mathbf{j}$

17 b When the train is moving NE, the coefficients of the **i** and **j** components are equal and positive.

$$t^2+3=3t+13$$
$$t^2-3t-10=0$$
$$(t-5)(t+2)=0$$
$$t=5,\ -2$$

Ignoring the negative root, as it denotes a time before the train was moving, the train is moving NE at $t = 5$ s (3s.f.).

Challenge

1 a $s(0)=20$ m

b $\frac{\mathrm{d}s}{\mathrm{d}t}=(20-t^2)\times\frac{1}{2}(t+1)^{-\frac{1}{2}}-2t(t+1)^{\frac{1}{2}}$

$$=\frac{(20-t^2)-4t(t+1)}{2(t+1)^{\frac{1}{2}}}$$

$$=\frac{20-4t-5t^2}{2\sqrt{t+1}}$$

Particle changes direction when $v=\frac{\mathrm{d}s}{\mathrm{d}t}=0\Rightarrow$

$$20-4t-5t^2=0$$

$t=1.64$ s (ignoring negative root, since $t\geqslant 0$)

So particle changes direction exactly once, when $t=1.64$ s

c Particle crosses O when $s=0$

$$0=(20-t^2)\sqrt{t+1}$$
$$t=\sqrt{20}$$

At $t=\sqrt{20}$ s, $\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{20-4\sqrt{20}-5\times 20}{2\sqrt{\sqrt{20}+1}}$

$$=\frac{-40-2\sqrt{20}}{\sqrt{\sqrt{20}+1}}$$

$$=-2\sqrt{20}\left(\sqrt{20}+1\right)^{\frac{1}{2}}$$

Challenge

2 a $\mathbf{v} = \dot{\mathbf{r}} = (6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j}$

$v^2 = |\mathbf{v}|^2 = 36\omega^2\cos^2\omega t + 16\omega^2\sin^2\omega t$

$= 36\omega^2\left(\frac{1}{2}+\frac{1}{2}\cos 2\omega t\right) + 16\omega^2\left(\frac{1}{2}-\frac{1}{2}\cos 2\omega t\right)$

$= 18\omega^2 + 18\omega^2\cos 2\omega t + 8\omega^2 - 8\omega^2\cos 2\omega t$

$= 26\omega^2 + 10\omega^2\cos 2\omega t$

$= 2\omega^2(13 + 5\cos 2\omega t)$, as required.

Use the double angle formulae $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$

b As $-1 \leqslant \cos 2\omega t \leqslant 1$

$2\omega^2(13-5) \leqslant 2\omega^2(13+5\cos 2\omega t) \leqslant 2\omega^2(13+5)$

$16\omega^2 \leqslant v^2 \leqslant 36\omega^2$

As $v > 0$ and $\omega > 0$, we can take the square root of each term and it will not change the inequality signs:

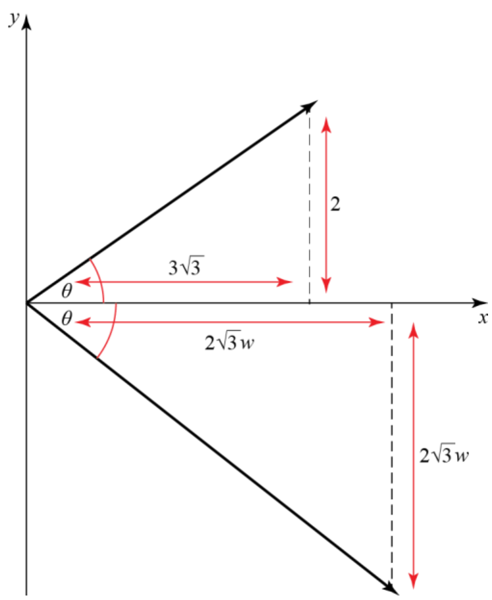
$4\omega \leqslant v \leqslant 6\omega$, as required.

c When $t = \frac{\pi}{3\omega}$

$\mathbf{r} = \left(6\sin\frac{\pi}{3}\right)\mathbf{i} + \left(4\cos\frac{\pi}{3}\right)\mathbf{j} = 3\sqrt{3}\,\mathbf{i} + 2\,\mathbf{j}$

$\dot{\mathbf{r}} = \left(6\omega\cos\frac{\pi}{3}\right)\mathbf{i} - \left(4\omega\sin\frac{\pi}{3}\right)\mathbf{j} = 3\omega\mathbf{i} - 2\sqrt{3}\omega\,\mathbf{j}$

Using $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



A diagram is essential here. Once the diagram has been drawn, the problem reduces to basic trigonometry. You find the angles using the inverse tangent button on your calculator.

$\tan\theta = \frac{2}{3\sqrt{3}} \Rightarrow \theta = 0.3674...^c$

$\tan\phi = \frac{2\sqrt{3}\omega}{3\omega} = \frac{2\sqrt{3}}{3} \Rightarrow \phi = 0.8570...^c$

The angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ is

$\theta + \phi = 0.3674...^c + 0.8570...^c = 1.224...^c = 70.2°$ (3 s.f.)