Exam-style practice: Paper 3, Section A: Statistics

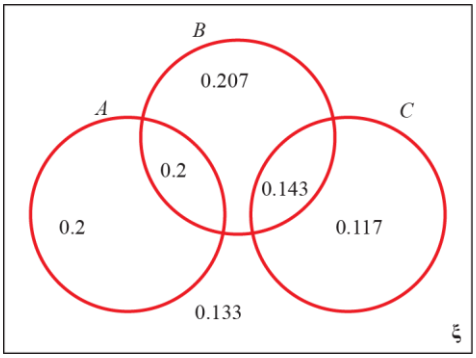
1 a Use the cumulative binomial distribution tables, with n = 40 and p = 0.52. Then $P(X \geqslant 22) = 1 - P(X \leqslant 21) = 1 - 0.5867 = 0.4133$ (4 s.f.).

b In order for the normal approximation to be used as an approximation to the binomial distribution the two conditions are: (i) n is large (>50); and (ii) p is close to 0.5.

c The two conditions for the normal approximation to be a valid approximation are satisfied. $\mu = np = 250 \times 0.52 = 130$ and $\sigma = \sqrt{np(1-p)} = \sqrt{130 \times 0.48} = \sqrt{62.4} = 7.90$ (3 s.f.). Therefore $B(250, 0.52) \approx N(130, 7\cdot9^2)$ so that $P(B \leqslant 120) \approx P(N \leqslant 120.5) = 0.1146$ (4 s.f.).

d If the engineer's claim is true, then the observed result had a less than 12% chance of occurring. This would mean that there would be insufficient evidence to reject her claim with a two-tailed hypothesis test at the 10% level. Though it does provide some doubt as to the validity of her claim.

2 a Since A and C are mutually exclusive, $P(A \cap C) = 0$ and their intersection need not be represented on the Venn diagram. Since B and C are independent, $P(B \cap C) = P(B) \times P(C) = 0.55 \times 0.26 = 0.143$. Using the remaining information in the question allows for the other regions to be labelled. Therefore the completed Venn diagram should be:



b $P(A) \times P(B) = 0.4 \times 0.55 = 0.22 \neq 0.2 = P(A \cap B)$ and so the events are not independent.

c $P(A \mid B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{0.2}{0.2 + 0.117 + 0.133} = \dfrac{0.2}{0.45} = 0.444$ (3 s.f.)

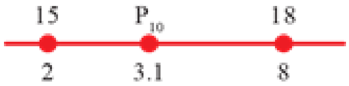
d $P(C \mid (A \cap B)') = \dfrac{P(C \cap (A \cap B)')}{P((A \cap B)')} = \dfrac{P(C)}{1 - P(A \cap B)} = \dfrac{0.26}{0.8} = 0.325$

3 a The variable t is continuous, since it can take any value between 12 and 26 (in degrees Celsius).

b Estimated mean 19.419; estimated standard deviation 2.814 (3 d.p.).

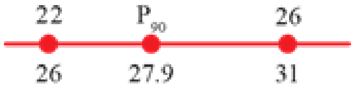
c Temperature is continuous and the data were given in a grouped frequency table.

3 d The 10th percentile is $\frac{31}{10} =$ 3.1th value. Using linear interpolation:



$$\frac{P_{10}-15}{18-15}=\frac{3.1-2}{8-2} \Rightarrow \frac{P_{10}-15}{3}=\frac{1.1}{6} \Rightarrow P_{10}=3\times\frac{1.1}{6}+15=0.55+15=15.5$$

The 90th percentile is $\frac{9\times 31}{10} =$ 27.9th value. Using linear interpolation:



$$\frac{P_{90}-22}{26-22}=\frac{27.9-26}{31-26} \Rightarrow \frac{P_{90}-22}{4}=\frac{1.9}{5} \Rightarrow P_{90}=4\times\frac{1.9}{5}+22=1.52+22=23.52$$

Therefore the 10th to 90th interpercentile range is $23.52-15.55=7.97$.

e Since the meteorologist believes that there is positive correlation, the hypotheses are

$H_0 : \rho = 0$

$H_1 : \rho > 0$

The sample size is 8, and so the critical value (for a one-tailed test) is 0.6215.
Since $r = 0.612 < 0.6215$, there is not sufficient evidence to reject H_0, and so there is not sufficient evidence, at the 5% significance level, to say that there is a positive correlation between the daily mean air temperature and the number of hours of sunshine.

4 a The value of 0.9998 is very close to 1, indicating that the plot of x against y is very close to being a linear relationship, and so the data should be well-modelled by an equation of the form $q = kt^n$.

b Rearranging the equation

$y = 0.07601 + 2.1317x$

$\Rightarrow \log q = 0.07601 + 2.1317 \log t$

$\Rightarrow q = 10^{0.07601+2.1317\log t} = 10^{0.07601} \times 10^{2.1317\log t}$

$\Rightarrow q = 10^{0.07601} \times 10^{\log t^{2.1317}} = 10^{0.07601} \times t^{2.1317}$

Therefore $k = 10^{0.0761} = 1.19$ (3 s.f.) and $n = 2.1317$.

c It would not be sensible to use the model to predict the amount of substance produced when $t = 85^{\circ}\text{C}$, since this is considerably outside the range of the provided data (extrapolation).

5 a $P(Z < a) = 0.025 \Rightarrow a = -1.96$ and $P(Z > a) = 0.05 \Rightarrow a = 1.645$. Therefore, for the given distribution, $\frac{3.416-\mu}{\sigma} = -1.96$ and $\frac{4.858-\mu}{\sigma} = 1.645$. Rearranging these equations:
$\frac{3.416-\mu}{\sigma} = -1.96 \Rightarrow 3.416-\mu = -1.96\sigma$ and $\frac{4.858-\mu}{\sigma} = 1.645 \Rightarrow 4.858-\mu = 1.645\sigma$.
Now subtract the second equation from the first to obtain:
$4.858-\mu-(3.416-\mu) = 1.645\sigma-(-1.96\sigma) \Rightarrow 1.442 = 3.605\sigma \Rightarrow \sigma = 0.4$ and so, using the first equation, $3.416-\mu = -1.96\times 0.4 \Rightarrow \mu = 3.416+0.784 = 4.2$. Using these values within the normal distribution, $P(3.5 < X < 4.6) = P(4.6) - P(3.5) = 0.84134 - 0.04006 = 0.8013$ (4 s.f.) of the cats will be of the standard weight.

b Using the binomial distribution, $P(B \geqslant 10) = 1 - P(B \leqslant 9) = 1 - 0.0594 = 0.9406$ (4 s.f.).

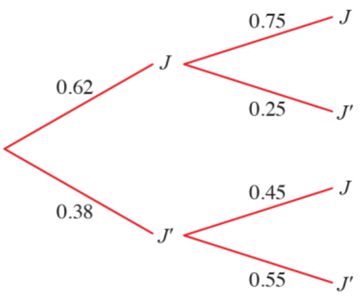
c Assume the mean is 4.5kg and standard deviation is 0.51. Then the sample $\bar{X}$ should be normally distributed with $\bar{X} \sim N\left(4.5, \frac{0.51^2}{12}\right)$. The hypothesis test should determine whether it is statistically significant, at the 10% level, that the mean is not 4.5kg. Therefore the test should be 2-tailed with

$H_0 : \mu = 4.5$
$H_1 : \mu \neq 4.5$

The critical region therefore consists of values greater than a where $P(\bar{X} > a) = 0.05$ and so $a = 4.742$ (4 s.f.) and values less than b where $P(\bar{X} < b) = 0.05$ and so $b = 4.258$ (4 s.f.).

Since the observed mean is 4.73 and 4.73 < a = 4.742, there is not enough evidence, at the 10% significance level, to reject H_0 i.e. there is not sufficient evidence to say, at the 10% level, that the mean weight of all cats in the town is different from 4.5kg.

6 It is first worth displaying the information in a tree diagram. Let J denote the event that Jemima wins a game of tennis and J' be the event that Jemima loses a game of tennis. Since Jemima either wins or loses each game of tennis, $P(J) + P(J') = 1$. This allows for the other probabilities on the tree diagram to be filled in. Therefore the completed tree diagram should be:



The required probability is then:
$P(\text{wins both games} \mid \text{wins second game})$

$$= \frac{P(\text{wins both games})}{P(\text{wins second game})} = \frac{0.62\times 0.75}{0.62\times 0.75 + 0.38\times 0.45} = \frac{0.465}{0.465+0.171} = 0.731 \text{ (3 s.f.)}$$

Exam-style practice: Paper 3, Section B: Mechanics

7 $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t$

$= \int (2-6t^2)\mathbf{i} - t\mathbf{j}\,\mathrm{d}t$

$= \left(2t - \frac{6}{3}t^3\right)\mathbf{i} - \frac{t^2}{2}\mathbf{j} + c$

At $t = 1$ s, $\mathbf{r} = 5\mathbf{i}$ m $\Rightarrow 5\mathbf{i} = (2-2)\mathbf{i} - \frac{1}{2}\mathbf{j} + c$

$c = 5\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\therefore \mathbf{r} = (2t - 2t^3 + 5)\mathbf{i} + \frac{1}{2}(1-t^2)\mathbf{j}$

When $t = 3$ s,

$\mathbf{r} = (6 - 54 + 5)\mathbf{i} + \frac{1}{2}(1-9)\mathbf{j}$

$\mathbf{r} = -43\mathbf{i} - 4\mathbf{j}$

$s = |\mathbf{r}| = \sqrt{43^2 + 4^2} = 43.185...$

At $t = 3$ s, P is 43.2 m from O (3 s.f.).

8 $R(\rightarrow)$: $u_x = 100\cos 30^\circ = 50\sqrt{3}$

$R(\uparrow)$: $u_y = 100\cos 30^\circ = 50$

a $R(\uparrow)$: $u_y = 50$ ms^{-1}, s = 0 m, $a = g = -9.8$ ms^{-2}, $t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 50t - 4.9t^2$

$4.9t^2 = 50t$

The solution $t = 0$ corresponds to the time the arrow is fired and can therefore be ignored.

$\therefore t = \frac{50}{4.9} = 10.204...$

The arrow reaches the ground after 10.2 s (3 s.f.).

b At maximum height, $v_y = 0$

$R(\uparrow)$: $u_y = 50$ ms^{-1}, $v_y = 0$ m, $a = g = -9.8$ ms^{-2}, $s = ?$

$v^2 = u^2 + 2as$

$0 = 50^2 - 19.6s$

$19.6s = 2500$

$s = \frac{2500}{19.6} = 127.55...$

The maximum height reached by the arrow is 128 m (3s.f.).

8 c At $t = 3$ s,

$R(\rightarrow)$: $v_x = u_x = 50\sqrt{3}$ ms^{-1} since horizontal speed remains constant.

$R(\uparrow)$: $u_y = 50$ ms^{-1}, $t = 3$ s, $a = g = -9.8$ ms^{-2}, $v_y = ?$

$$v = u + at$$

$$v_y = 50 - (3\times 9.8) = 20.6$$

The speed at $t = 3$ s is given by:

$$v^2 = v_x^2 + v_y^2$$

$$v^2 = \left(50\sqrt{3}\right)^2 + \left(20.6\right)^2$$

$$v = \sqrt{7500 + 424.36} = 89.018...$$

The speed of the arrow after 3 s is 89.0 ms^{-1} (3 s.f).

9 a $\mathbf{u} = 2\mathbf{i}$ ms^{-1}, $t = 10$ s, $\mathbf{a} = 0.2\mathbf{i} - 0.8\mathbf{j}$ ms^{-2}, $\mathbf{r} = ?$

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = 20\mathbf{i} + \frac{100}{2}(0.2\mathbf{i} - 0.8\mathbf{j})$$

$$\mathbf{r} = 20\mathbf{i} + 10\mathbf{i} - 40\mathbf{j}$$

After 10 s, the position vector of the cyclist is $(30\mathbf{i} - 40\mathbf{j})$ m.

b $s = |\mathbf{r}|$

$$s = \sqrt{30^2 + 40^2} = 50$$

After 10 s, the cyclist is 50 m from A.

c For t >10 s, $\mathbf{v} = 5\mathbf{i}$ ms^{-1} and $\mathbf{a} = 0$

The position vector is now given by:

$$\mathbf{r} = (30\mathbf{i} - 40\mathbf{j}) + \mathbf{v}(t - 10)\mathbf{i}$$

$$\mathbf{r} = 30\mathbf{i} - 40\mathbf{j} + 5(t - 10)\mathbf{i}$$

$$\mathbf{r} = (5t - 20)\mathbf{i} - 40\mathbf{j}$$

The cyclist will be south-east of A when the coefficient of **i** is positive and coefficient of **j** is negative, but both have equal magnitude.

$$5t - 20 = 40$$

$$5t = 60$$

$$t = \frac{60}{5} = 12$$

The cyclist is directly south-east of A after 12 s.

9 d First, work out the position vector of B from A:

$\mathbf{r} = (5t - 20)\mathbf{i} - 40\mathbf{j}$

Cyclist reaches B when $t = 12 + 30 = 42$ s

$\mathbf{r} = ((5 \times 42) - 20)\mathbf{i} - 40\mathbf{j}$

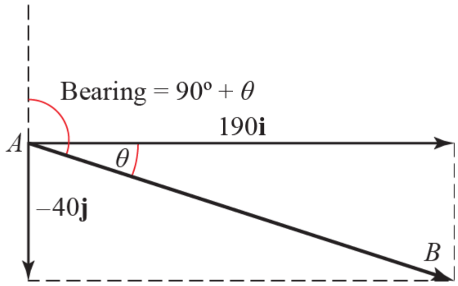
$\mathbf{r} = 190\mathbf{i} - 40\mathbf{j}$

Let θ be the acute angle between the horizontal and B (as shown in the diagram).

Then $\tan\theta = \frac{40}{190}$

$\theta = 11.888...$

To the nearest degree, the bearing of B from A is 90 +12 = 102°.



10 a Considering Q and using Newton's second law of motion:

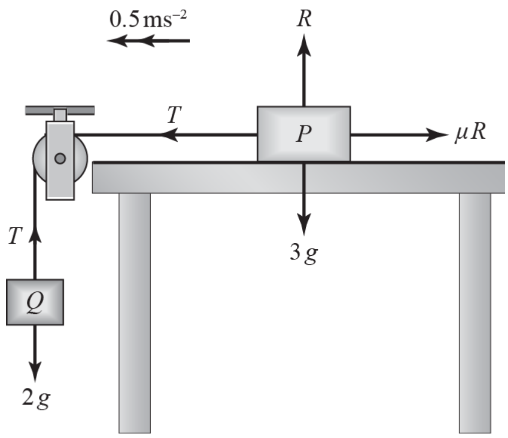
$a = 0.5$ ms^{-2}, $m = 2$ kg

$F = ma$

$2g - T = 2 \times 0.5$

$T = (2 \times 9.8) - 1 = 18.6$

The tension in the string immediately after the particles begin to move is 18.6 N.



b Considering P:

Resolving vertically $\Rightarrow R = 3g$

Resolving horizontally and using Newton's second law of motion with $a = 0.5$ ms^{-2} and $m = 3$ kg:

$T - \mu R = 3 \times 0.5$

$3\mu g = T - 1.5$

$\mu = \frac{18.6 - 1.5}{3 \times 9.8} = 0.58163...$

The coefficient of friction is 0.582 (3 s.f.), as required.

10 c Consider P before string breaks: $u = 0\text{ ms}^{-1}$, $t = 2$ s, $a = 0.5\text{ ms}^{-2}$, $v = ?$

$$v = u + at$$
$$v = 0 + (0.5 \times 2) = 1$$

After string breaks, the only force acting on P is a frictional force of magnitude $F = \mu R = 3\mu g$

Using Newton's Second Law for P,

$$F = ma$$
$$3\mu g = 3a$$
$$a = \mu g$$
$$a = 9.8 \times 0.58163...$$
$$= 5.7$$

The acceleration is in the opposite direction to the motion of P, hence

$u = 1\text{ ms}^{-1}$, $v = 0\text{ ms}^{-1}$, $a = -0.5\text{ ms}^{-2}$, $t = ?$

$$v = u + at$$
$$0 = 1 - 5.7t$$
$$t = \frac{1}{5.7} = 0.17543...$$

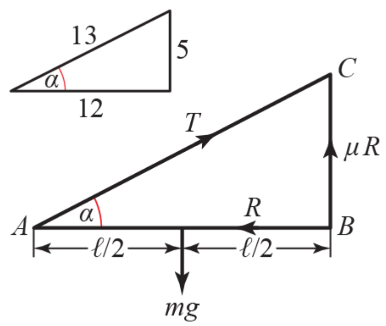
P takes 0.175 s (3 s.f.) to come to rest.

d The information that the string is inextensible has been used in assuming that the tension is the same in all parts of the string and that the acceleration of P and Q are identical while they are connected.

11 a The rod is in equilibrium so resultant force and moment are both zero.

$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$

Taking moments about B:



$$mg\frac{l}{2} = (T\sin\alpha) \times l$$
$$T = \frac{mg}{2\sin\alpha}$$
$$T = \frac{mg}{2 \times \frac{5}{13}} = \frac{13mg}{10} \quad \text{as required.}$$

b Resolving horizontally:

$$R = T\cos\alpha$$
$$R = \frac{13mg}{10} \times \frac{12}{13} = \frac{6mg}{5}$$

Resolving vertically:

$$T\sin\alpha + \mu R = mg$$
$$\left(\frac{13mg}{10} \times \frac{5}{13}\right) + \mu\frac{6mg}{5} = mg$$
$$\frac{6}{5}\mu = 1 - \frac{1}{2}$$
$$\mu = \frac{5}{12}$$

The coefficient of friction between the rod and the wall is $\frac{5}{12}$.